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OUTSTANDING SCHOLARSHIP EXEMPLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
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Scholarship 2021
Calculus

Time allowed: Three hours
Total score: 40

ANSWER BOOKLET

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write your answers in this booklet.

Make sure that you have Formulae Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	

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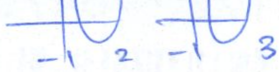

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1a) $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x - 3}$, $x \in \mathbb{R}$. $f(x) > 0$.

$= \dfrac{(x+1)(x-2)}{(x+1)(x-3)}$

Critical regions: $x < -1$, $-1 < x < 2$, $2 < x < 3$, $x > 3$

For $x < -1$, $x^2 - x - 2 > 0$ and $x^2 - 2x - 3 > 0$

$\therefore f(x) > 0$

For $-1 < x < 2$, $x^2 - x - 2 < 0$ and $x^2 - 2x - 3 < 0$

$\therefore f(x) > 0$

For $2 < x < 3$, $x^2 - x - 2 > 0$ and $x^2 - 2x - 3 < 0$

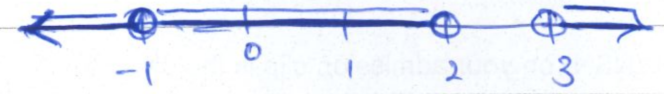
$\therefore f(x) < 0$

For $x > 3$, $x^2 - x - 2 > 0$ and $x^2 - 2x - 3 > 0$

$\therefore f(x) > 0$

$\therefore f(x) > 0$, $x < -1$, $-1 < x < 2$, $x > 3$

Number Line:



$\Longrightarrow$ arrows represent regions where $f(x) > 0$.

1b) $x^{x\sqrt{x}} = x^{2x}$, where $x > 0$ as $\sqrt{x}$ only $\mathbb{R}$ when $x > 0$ and $x \neq 0$ as 0^0 is undefined.

$\therefore x\sqrt{x} \ln x - 2x \ln x = 0$

$(x\sqrt{x} - 2x) \ln x = 0$

~~$\therefore$ ... $\neq 0$ as ...~~

~~$\ln x = 0 \Rightarrow x = 1$~~

~~$x - 2 = 0 \Rightarrow x = 2$.~~

$\ln x = 0 \Rightarrow x = 1$

$x\sqrt{x} - 2x = 0$

$x(\sqrt{x} - 2) = 0$

$x \neq 0$ $\therefore \sqrt{x} = 2 \Rightarrow x = 4$

$\therefore x = 1, 4$

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1c) $y = 2x^2 - x - 1$ & $y = -2x^2 - x + 1$

$\therefore\ 2x^2 \cancel{-x} - 1 = -2x^2 \cancel{-x} + 1$

$\Rightarrow\ 4x^2 - 2 = 0$

$\Rightarrow\ 2x^2 - 1 = 0$

$\Rightarrow\ x = \pm\frac{1}{\sqrt{2}}$

$(-2x^2 - \cancel{x} + 1 - 2x^2 + \cancel{x} + 1)$
$= (-4x^2 + 2)$

$\therefore$ Let $x = a$ be the equation of the vertical line.

$\therefore\ 2\int_{-\frac{1}{\sqrt2}}^{a} -4x^2 + 2\,dx = \int_{-\frac{1}{\sqrt2}}^{+\frac{1}{\sqrt2}} -4x^2 + 2\,dx$

$\Rightarrow 2\left[-\frac{4x^3}{3} + 2x\right]_{-\frac{1}{\sqrt2}}^{a} = \left[-\frac{4x^3}{3} + 2x\right]_{-\frac{1}{\sqrt2}}^{\frac{1}{\sqrt2}}$

$\Rightarrow 2\left[-\frac{4a^3}{3} + 2a - \left[-\frac{4\left(-\frac{1}{\sqrt2}\right)^3}{3} + 2\left(-\frac{1}{\sqrt2}\right)\right]\right]$

$= \left[-\frac{4\left(\frac{1}{\sqrt2}\right)^3}{3} + 2\left(\frac{1}{\sqrt2}\right) - \left[+\frac{4}{3}\left(+\frac{1}{\sqrt2}\right)^3 + 2\left(-\frac{1}{\sqrt2}\right)\right]\right]$

$\Rightarrow 2\left[-\frac{4a^3}{3} + 2a - \left[\frac{4}{3\cdot 2\sqrt2} - \sqrt2\right]\right]$

$= -\frac{8}{3}\left(\frac{1}{\sqrt2}\right)^3 + \frac{2\cdot 2}{\sqrt2}$

$\Rightarrow -\frac{4a^3}{3} + 2a = \cancel{-\frac{2}{3}\left(\frac{1}{\sqrt2}\right)^3} + \cancel{\frac{2}{\sqrt2}} + \cancel{\frac{2}{3\sqrt2}} - \cancel{\sqrt2}$

$\Rightarrow -\frac{4a^3}{3} + 2a = 0 \Rightarrow a\left(-\frac{4a^2}{3} + 2\right) = 0$

$a = 0$, or $-\frac{4a^2}{3} = -2 \Rightarrow a^2 = \frac{3}{4} \Rightarrow a = \pm\frac{\sqrt3}{2}$

But $a \in \left(-\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$ $\therefore a \neq \pm\frac{\sqrt3}{2}$ $\therefore a = 0$ $\therefore x = 0$ is the vertical line.

1d) $\int_0^2 \frac{x}{\sqrt{x+1}}\,dx$ $u=\sqrt{x+1} \Rightarrow x+1=u^2$

$\Rightarrow x=u^2-1$

~~x(0) = -1~~ $u(0)=1$

~~x(2) = 8~~ $u(2)=\sqrt{3}$

~~dx = 2u du~~

$dx = 2u\,du$

$x = u^2-1$

$\therefore \int_1^{\sqrt{3}} \frac{u^2-1}{\cancel{u}} \cdot 2\cancel{u}\,du$

$= 2\left[\frac{1}{3}u^3 - u\right]_1^{\sqrt{3}}$

$= 2\left[\frac{1}{3}\cdot\sqrt{3}^3 - \sqrt{3} - \left(\frac{1}{3}\cdot(1)^3 - (1)\right)\right]$

$= \frac{4}{3}$

1e) $y = |\sin x - \cos x|$, $x \in [0, 2\pi]$.

$x = 0, 2\pi$.

$\int_0^{2\pi} |\sin x - \cos x|\,dx = A$

$y(0) = y(2\pi) = 1$.

$|\sin x - \cos x| = 0$

$\therefore \sin x - \cos x = 0$

$\Leftrightarrow \sin x - \sin\left(\frac{\pi}{2} - x\right) = 0$

$\Leftrightarrow$ ~~2cos~~ $2\cos\left(\cancel{\frac{x+\frac{\pi}{2}-x}{2}}\right)\sin\left(\frac{x-\frac{\pi}{2}+x}{2}\right) = 0$

$\therefore \sin\left(x - \frac{\pi}{4}\right) = 0 = \sin 0 \;\therefore\; x = n\pi + (-1)^n \cdot 0 + \frac{\pi}{4}$

$= n\pi + \frac{\pi}{4}$, $n \in \mathbb{Z}$.

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$, for $x \in [0, 2\pi]$.

$\therefore \frac{A}{2} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} |\sin x - \cos x|\,dx$ ~~= B + ... D~~

~~1e continued)~~ $B = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \, dx$

$= -\left[\cos x + \sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \cancel{-1 + \frac{2}{\sqrt{2}}} - \left(1 - 2\times\frac{1}{\sqrt{2}}\right)$

$= \sqrt{2} - 1$

$C = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}}$

1e continued) $\frac{A}{2} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} |\sin x - \cos x| \, dx$

$\therefore \frac{A}{4} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} |\sin x - \cos x| \, dx$

$= B + C$

$B = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \, dx = -\left[\cos x + \sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \sqrt{2} - 1$

$C = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} |\sin x - \cos x| \, dx = -\left[\cos x + \sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}$

$= 1$

$\therefore A = 4(B + C) =$

$4\sqrt{2}$ units^2

2a) $\log_{\frac{a}{b}} b = 5$, $a, b > 0$ $\quad \log_{\frac{a}{b}} (\sqrt[3]{b} \times \sqrt[4]{a})$

$= \dfrac{\ln b}{\ln(\frac{a}{b})} = 5 \Rightarrow \dfrac{\ln b}{\ln a - \ln b} = 5 \Rightarrow \ln b = 5\ln a - 5\ln b$

$\Rightarrow \dfrac{6\ln b}{5} = \ln a$

$\therefore k = \log_{\frac{a}{b}} (\sqrt[3]{b} \sqrt[4]{a})$

$\therefore k = \dfrac{\ln \sqrt[3]{b} + \ln \sqrt[4]{a}}{\ln a - \ln b} = \dfrac{\frac{1}{3}\ln b + \frac{1}{4}\ln a}{\ln a - \ln b}$

$= \dfrac{\frac{1}{3}\ln b + \frac{1}{4}\cdot \frac{6}{5}\ln b}{\frac{6}{5}\ln b - \ln b}$

$= \dfrac{19}{6}$ $\quad = \dfrac{\frac{7}{6}\ln b}{\frac{1}{5}\ln b}$

2b) pairs $(x, y) \in Q^+$

where $x + y = 11$

$A = (x^2)(y^3) = (11-y)^2 y^3$

$= (121 - 22y + y^2) y^3$

$= 121y^3 - 22y^4 + y^5$

$\dfrac{dA}{dy} = 363y^2 - 88y^3 + 5y^4$

$= 0$

$\therefore y^2 (363 - 88y + 5y^2) = 0$

$\therefore y \neq 0$ as y is positive $\quad y = \dfrac{-(-88) \pm 22}{10}$

$= \dfrac{88 \pm 22}{10} = 11, 6.6$

$\dfrac{d^2A}{dy^2} = 726y - 264y^2 + 20y^3$

$\left.\dfrac{d^2A}{dy^2}\right|_{y=11} = 2662 > 0$ $\therefore$ minimum (concave up) and $x \neq 0$

$\left.\dfrac{d^2A}{dy^2}\right|_{y=6.6} = -958.32 < 0$ $\therefore$ maximum (concave down)

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2b continued) $\therefore$ $y = 6.6$ and $x = 4.4$

2c) $f(x) = a\sin(\pi x + \alpha) + b\cos(\pi x + \alpha) + 1$

$f(2020) = 10$ want $f(2021)$.

$f(2020) = a\sin[2020\pi + \alpha] + b\cos[2020\pi + \alpha] + 1$
$= 10$

$\therefore a\sin[2020\pi + \alpha] + b\cos[2020\pi + \alpha] = 9$

$a[\sin 2020\pi\cos\alpha + \sin\alpha\cos 2020\pi] + b[\cos 2020\pi\cos\alpha - \sin 2020\pi\sin\alpha] = 9$

$\Rightarrow a[\sin\alpha] + b[\cos\alpha] = 9$

$f(2021) = a\sin(2021\pi + \alpha) + b\cos(2021\pi + \alpha) + 1$
$= k$

$\therefore a[\sin 2021\pi\cos\alpha + \cos 2021\pi\sin\alpha]$
$+ b[\cos 2021\pi\cos\alpha - \sin 2021\pi\sin\alpha] = k - 1$

$\therefore a[-\sin\alpha] + b[-\cos\alpha] = k - 1$

$k - 1 = -[a\sin\alpha + b\cos\alpha] = -(9)$

$\therefore k = -8 = f(2021)$

2d) $f(x) = (x^2+1)^{\sin x}$ $f'(\frac{\pi}{2})$

$\ln(f(x)) = \sin x \cdot \ln(x^2+1) = \ln y$

$\therefore \frac{1}{y}\cdot\frac{dy}{dx} = \sin x \cdot \frac{2x}{x^2+1} + \cos x \cdot \ln(x^2+1)$

$\therefore \frac{dy}{dx} = \cancel{\left[\left(\frac{\pi}{2}\right)^2+1\right]^{\sin\frac{\pi}{2}}}\left[\cancel{\sin\frac{\pi}{2}} \cdot \frac{\frac{2\pi}{2}}{\cancel{\frac{\pi^2}{4}+1}}\right] = \pi.$

~~2c)~~ $f(x) = (\log_2 x)^2 + 6m(\log_2 x) + n$

$m, n \in \mathbb{R}$.

Local minimum $\left(\frac{1}{8}, -2\right)$

$\cancel{f'(x) =}$ $v = \log_2 x = \frac{\ln x}{\ln 2}$ $\therefore \frac{dv}{dx} = \frac{1}{x \ln 2}$

$\therefore f'(x) = 2 \cdot \frac{(\log_2 x)}{x \ln 2} + \frac{6m}{x \ln 2} + n$

~~and f''~~

$\therefore f(x) = \left(\log_2 \frac{1}{8}\right)^2 + 6m\left(\log_2\left(\frac{1}{8}\right)\right) + n = -2$

$\therefore \frac{1}{9} + 6m \cdot -\frac{1}{\cancel{3}} + n = -2$

$\therefore \frac{1}{9} - 2m + n = 2 \Rightarrow -2m + n = \frac{17}{9}$

$\Rightarrow m = \left(\frac{17}{9} - n\right) \cdot -\frac{1}{2}$

$\Sigma 1 \; f'\left(\frac{1}{8}\right) = 0$

$\therefore \cancel{(\log_2} \quad \frac{2\left(\log_2 \frac{1}{8}\right)}{x \ln 2} + \frac{6m}{x \ln 2} + n = 0$

$\therefore \frac{2\left(-\frac{1}{3}\right)}{\frac{1}{8}\ln 2} + \frac{6m}{\frac{1}{8}\ln 2} + n = 0$

$\therefore \begin{cases} -\frac{2}{3} + 6m + \frac{n \ln 2}{8} = 0 \\ \frac{3}{9} - 6m + 3n = 6 \end{cases}$

$-\frac{1}{3} + n\left(\frac{\ln 2 + 3}{8}\right) = 6$

$\Rightarrow n = \frac{8}{(\ln 2 + 3)} \cdot \frac{19}{3} = \frac{152}{3(3 + \ln 2)}$

$m = \left(\frac{17}{9} - \frac{152}{3(3 + \ln 2)}\right) \cdot -\frac{1}{2} = \frac{-51(3 + \ln 2) + 1368}{54(3 + \ln 2)}$

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3a) $ax^2+bx+c=0$ $\quad x=\sin\theta, \cos\theta$.

$-\frac{b}{a}$ = sum of roots

$\therefore -\frac{b}{a} = \sin\theta+\cos\theta = \frac{\sin\theta}{1-\cot\theta} + \frac{\cos\theta}{1-\tan\theta}$

$\therefore$ ~~RHS~~ $= \frac{\sin\theta\left(1-\frac{\sin\theta}{\cos\theta}\right)+\cos\theta\left(1-\frac{\cos\theta}{\sin\theta}\right)}{(1-\cot\theta)(1-\tan\theta)}$

$= \frac{\sin\theta-\frac{\sin^2\theta}{\cos\theta}+\cos\theta-\frac{\cos^2\theta}{\sin\theta}}{1-\tan\theta-\cot\theta+1}$

$= \frac{(\sin\theta+\cos\theta)-\left(\frac{\sin^2\theta}{\cos\theta}+\frac{\cos^2\theta}{\sin\theta}\right)}{2-\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)}$

$= \frac{\sin\theta+\cos\theta-\frac{\sin^3\theta+\cos^3\theta}{\sin\theta\cos\theta}}{~~2\sin\theta\cos\theta-1~~}$

~~$\cos\theta+\sin\theta$~~ $= \frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta}$

$= \frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}} = \frac{\sin^2\theta}{\sin\theta-\cos\theta} - \frac{\cos^2\theta}{-\cos\theta+\sin\theta}$.

$= \frac{\sin^2\theta-\cos^2\theta}{(\sin^2-\cos\theta)^2} = -\frac{(\sin\theta+\cos\theta)~~(\sin\theta-\cos\theta)~~}{(\sin\theta-\cos\theta)^2}$

Continued on page 19).

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~~3a) $\sin\theta + \cos\theta = \frac{\sin\theta}{1-\cot\theta} + \frac{\cos\theta}{1-\tan\theta}$~~

3b) $y = mx + 2\sqrt{21}$ $\quad$ $16x^2 - 9y^2 = 144$

Tangential.

$32x - 18y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = m$

$\frac{dy}{dx} = \frac{-32x}{-18y} = \frac{16x}{9y}$

$\therefore \frac{16x}{9y} = m \quad \therefore x = \frac{9my}{16}$

$\therefore y = \frac{9m^2y}{16} + 2\sqrt{21}$ $\quad$ and $\left(\left(\frac{9my}{16}\right)^2\right) - 9y^2 = 144$

or $\therefore y\left(1 - \frac{9m^2}{16}\right) = 2\sqrt{21}$

& $y^2\left(\frac{81m^2}{16} - 9\right) = 144$

$\therefore \frac{(2\sqrt{21})^2}{\left(1-\frac{9m^2}{16}\right)^2} = \frac{144}{\left(\frac{81m^2}{16} - 9\right)}$

$\therefore 84\left(\frac{81m^2}{16} - 9\right) = 144\left(1 - \frac{9m^2}{8} + \frac{81m^4}{256}\right)$

~~or $\frac{1701}{4}m^2 \ldots \frac{625m^2}{16} + 160$~~

or $\frac{729}{16}m^4 - \frac{2349}{4}m^2 + 900 = 0$

or $m^2 = \frac{\frac{2349}{4} \pm 425.25}{\frac{729}{8}} = \frac{100}{9}, \frac{16}{9}$

$\therefore m = \pm\frac{10}{3}, \pm\frac{4}{3}$ (4 values).

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3c) $f(x) = ax^3 - bx$, $a, b > 0$ and $a, b \in \mathbb{R}$.

$(0,0)$. $P(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$.

Want $f'(0)$ if

$m = \tan\theta$.

$f(0) = 0$

$f(-\sqrt{3}) = a(-\sqrt{3})^3 - b(-\sqrt{3}) = -3a\sqrt{3} + b\sqrt{3} = 0$

$f(\sqrt{3}) = a(\sqrt{3})^3 - b(\sqrt{3}) = 3a\sqrt{3} - b\sqrt{3} = 0$.

~~$f(x) = 0$~~

$f'(x) = 3ax^2 - b$

$f'(\sqrt{3}) = 9a - b = f'(-\sqrt{3})$

$= \tan 45° = 1$

$\therefore$ ~~$b = 9a$~~ $b = 9a - 1 \Rightarrow -b = 1 - 9a$

$\therefore f'(0) = -b$.

~~$3a\sqrt{3} - 9a\sqrt{3} = 0$~~

~~$3a\sqrt{3} - b\sqrt{3} = 0$~~

~~$9a - b = 0$~~

$\Rightarrow$ ~~$9a\sqrt{3} - b\sqrt{3} = 0$~~

$3a\sqrt{3} - (9a - 1)\sqrt{3} = 0$

$\Rightarrow 3a\sqrt{3} - 9a\sqrt{3} + \sqrt{3} = 0$

$\Rightarrow a(3\sqrt{3} - 9\sqrt{3}) = -\sqrt{3}$

$\Rightarrow a = \dfrac{\sqrt{3}}{9\sqrt{3} - 3\sqrt{3}} = \dfrac{1}{6}$

$\therefore f'(0) = -\dfrac{1}{2}$

3di) Permutations count. $\therefore\ 5! = {}^5P_5 = 120$ ways.

3dii) seven children, 2 girls next to each other.

$K = $ ~~[4 5][4]~~ $\quad {}^5P_1 \times {}^4P_1 \times {}^3P_1 \times {}^2P_1 \times {}^1P_1 \times 6 \times 2$ stand next to each other

$= 1440$ ways.

3diii) All permutations $=$ $\quad {}^7P_7 = 5040$

~~$\therefore$ P(never stand next to each other) $=$~~

~~$1 - k$~~

$\therefore$ Never next to each other $= 5040 - 1440$

$= 3600$ ways.

4a) 0.16 is the average compounding growth ~~year~~ rate.

$D = 4500 + 500 = 5000$

$$\frac{dA}{dt} = 0.16A + 5000$$

$$\therefore \int \frac{0.16}{0.16A + 5000}\,dA = \int 0.16\,dt$$

$$\Rightarrow e^{\ln|k(0.16A + 5000)|} = e^{0.16t}$$

$$\Rightarrow 0.16A + 5000 = Be^{0.16t}$$

$$\Rightarrow 0.16(76000) + 5000 = B$$

$$\Rightarrow B = 17160$$

$$\therefore A = \frac{17160e^{0.16t} - 5000}{0.16}$$

$t = 10$

$$\therefore A = \frac{17160e^{0.16\times 10} - 5000}{0.16} = \$499962.73 \text{ (2dp)}$$

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4a continued). Therefore, they will be unable to ~~meet~~ ^than the needed^ \$500,000 by ~~to~~ falling \$ 37.27 short in 10 years, so they will need to increase the initial investment.

4bi) $y = f(x)$

$$\frac{dy}{dx} = (x-1)y^3 \qquad f(0) = a,\ a > 0$$

$$\therefore \int y^{-3}\,dy = \int (x-1)\,dx$$

$$-\frac{1}{2}y^{-2} \ \cancel{dx} = \frac{1}{2}x^2 - x + c.$$

$$\therefore -\frac{1}{2a^2} = c \quad \therefore +\frac{1}{2}y^{-2} = -\frac{1}{2}x^2 + x + \frac{1}{2a^2}$$

$$\Leftrightarrow y^{-2} = -x^2 + 2x + \frac{1}{a^2} = \frac{1}{y^2}$$

$$\Leftrightarrow y = \cancel{\pm}+\frac{1}{\sqrt{-x^2+2x+\frac{1}{a^2}}} \qquad -x^2+2x+\frac{1}{a^2} > 0$$

only positive as $f(0) = a$.

4bii) $-x^2 + 2x + \frac{1}{a^2} > 0 \Rightarrow x^2 - 2x - \frac{1}{a^2} < 0$

Boundary: $x^2 - 2x - \frac{1}{a^2} = 0$

$$\Leftrightarrow a^2x^2 - 2a^2x - 1 = 0$$

$$\cancel{\text{Let}} \quad x = \frac{2a^2 \pm \sqrt{4a^4 + 4a^2}}{2a^2}$$

$$= \frac{2a^2 \pm 2a\sqrt{a^2+1}}{2a^2}$$

not including boundary

$$\therefore \text{Domain: } \left(\frac{a - \sqrt{a^2+1}}{a}, \frac{a+\sqrt{a^2+1}}{a}\right) = \frac{a \pm \sqrt{a^2+1}}{a}, \quad a > 0$$

$x = 1$ gives ~~max~~ minimum $\Rightarrow y = \pm \dfrac{1}{\sqrt{1+\frac{1}{a^2}}}$ ~~maximum~~ minimum.

4b ii) continued) ~~maximum~~ at $y = \frac{a \pm \sqrt{a^2+1}}{a}$

as it is the restricted domain's upper & lower limit - not maximum - asymptote

$\therefore$ Range is $\left[\frac{1}{\sqrt{1+\frac{1}{a^2}}}, \infty\right)$ 0

~~and $\left[-\frac{1}{\sqrt{1+a^2}}, -\infty\right)$ ← for other branch.~~

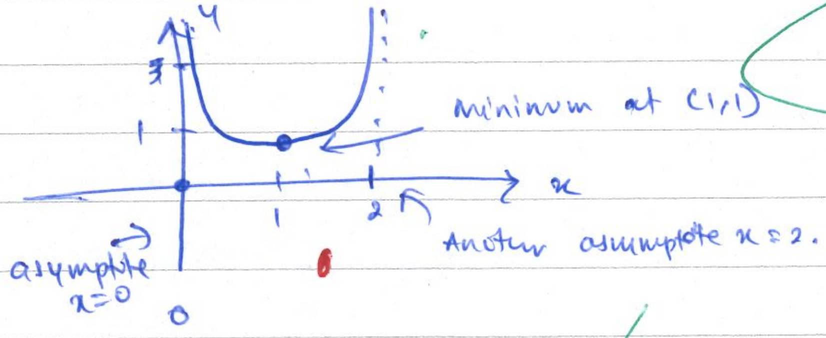
4b iii) $a \to +\infty$ Domain is $\left(\frac{a-\sqrt{a^2+1}}{a}, \frac{a+\sqrt{a^2+1}}{a}\right)$

$$\lim_{a \to +\infty} \left(\frac{a}{a} \pm \sqrt{\frac{a^2}{a^2} + \frac{1}{a^2}}\right)$$

$$= 1 \pm \sqrt{1} = 1 \pm 1$$

$$= 0, 2.$$

$\therefore$



Minimum $= \lim_{a \to +\infty} \frac{1}{\sqrt{1+\frac{1}{a^2}}} = \frac{1}{1} = 1$

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4c) $T_n = \sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}}$ $n \in \mathbb{N}$.

~~$T_1 = \sqrt{1+1+}$~~

n	T_n	S_n
1	$\frac{3}{2}$	$\frac{3}{2}$
2	$\frac{7}{6}$	$\frac{8}{3}$
3	$\frac{13}{12}$	$\frac{15}{4}$
4	$\frac{21}{20}$	$\frac{24}{5}$
5	$\frac{31}{30}$	$\frac{35}{6}$
6	$\frac{43}{42}$	$\frac{48}{7}$
7	$\frac{57}{56}$	
⋮	⋮	

~~denominator + 1 / denominator.~~

$S_n = \frac{\text{numerator}}{n+1}$

n	numerator
1	3 +5
2	8 +7)+2
3	15 +9)+2
4	24 +11)+2
5	35
⋮	⋮

~~n~~	~~Tn (numerator) = Tn(num)~~		
1	3	+4	
2	7	+6	+2
3	13	+8	+2
4	21	+10	+2
5	31	+12	+2
6	43	+14	+2
7	57		

∴ $T_n(\text{num}) = n^2 + bn + c$

$c = 1$

∴ $b = 1$

∴ $n^2 + n + 1$

∴ ~~Denomi~~

$T_n = \frac{n^2+n+1}{n^2+n}$

∴ numerator $= an^2 + bn + c$ $a = \frac{2}{2} = 1$

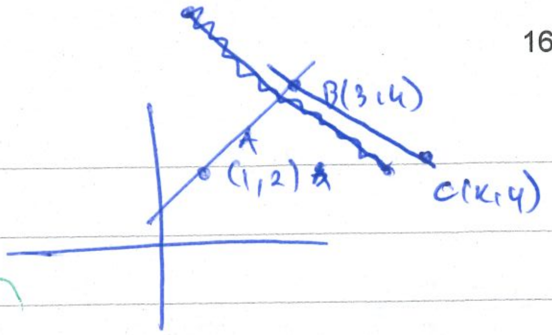
$c = 0$

∴ $b = 2$ ∴ numerator $= n^2 + 2n$.

∴ $S_n = \frac{n^2+2n}{n+1}$

∴ $S_{2021} = \sum_{n=1}^{2021} T_n = \frac{2021(2023)}{2022}$ ∴ $\frac{4088483}{2022}$

5a)



BC $\perp$ AB.

$m(AB) = \frac{4-2}{3-1} = 1$

$\therefore m(BC) = -1$ as $m m_{\perp} = -$

$|BC| = 4|AB|$

$\cancel{AB} = |AB| = \sqrt{(3-1)^2 + (4-2)^2}$

$= \sqrt{8}$

$= 2\sqrt{2}$

$\therefore |BC| = 4 \cdot 2\sqrt{2}$

$= 8\sqrt{2}.$

$\therefore (8\sqrt{2})^2 = (3-x)^2 + (4-y)^2$

$\frac{4-y}{3-x} = -1 \Rightarrow 4-y = x-3$

$y = 4-(x-3)$

$\therefore 4-y = (3-x)\times -1$

$\therefore 128 = 2(3-x)^2$

$\Rightarrow 3-x = \pm 8$

$\Rightarrow x = -5, 11$

$\therefore y = 12, -4$

$\therefore (-5, 12)$ & $(11, -4)$

are $C(x,y)$.

5b) $x, y \in \mathbb{R}$. $z = x+iy$.

$z + \frac{1}{\bar{z}} = \bar{z} + \frac{1}{z}$

$\therefore x+iy + \frac{1}{x-iy} = x-iy + \frac{1}{x+iy}$

$2iy = \frac{x-iy - x-iy}{x^2+y^2} = \frac{-2iy}{x^2+y^2}$

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5b continued) $\therefore \quad x^2+y^2=-1$

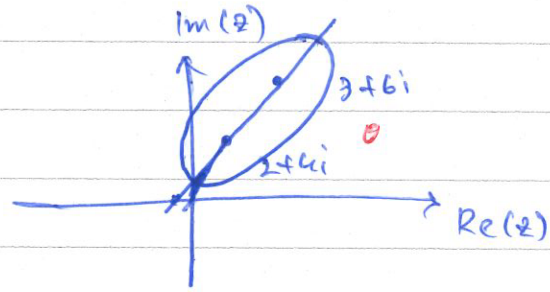
But $x, y \in \mathbb{R}$. $\therefore$ $\alpha \in \mathbb{R}$ then $x^2<0$ is not possible,

so no such z exist.

5ci) $|z-(2+4i)|+|z-(3+6i)|=4$

Ellipse : $2+4i$ and $3+6i$ foci.

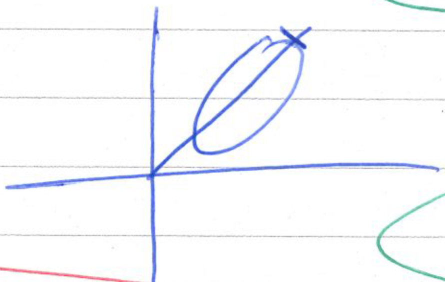
~~Distance between~~



5cii) Maximum $|z|$ ellipse

$m = \dfrac{6-4}{3-2} = 2$

$\therefore \quad y = 2x+2$



5d) $(z_2-z_1)(z_1-z_3) = \left(\dfrac{1}{2}(z_2-z_3)\sec\alpha\right)^2$

$\therefore \quad = \dfrac{1}{4}(z_2-z_3)^2\sec^2\alpha$

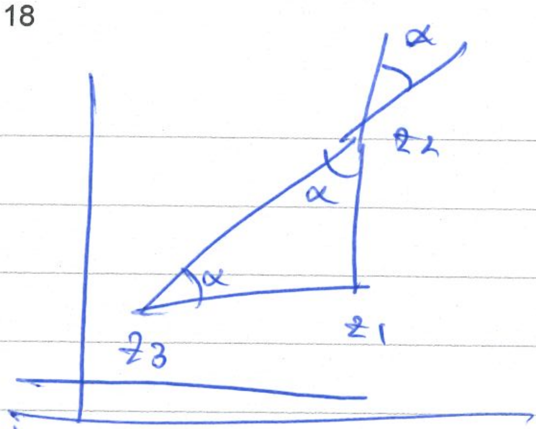
constant ~~arg~~ argument $\rightarrow$ $\therefore \arg\left(\dfrac{1}{4}\sec^2\alpha\right) = \arg\left[\dfrac{(z_2-z_1)}{(z_2-z_3)}\right] + \arg\left[\dfrac{(z_1-z_3)}{(z_2-z_3)}\right] = 0$

$\arg\left(\dfrac{-z_2+z_1}{-z_2+z_3}\right) = -\alpha$ (opposite side) and $\arg\left(\dfrac{z_1-z_3}{z_2-z_3}\right) = \alpha$

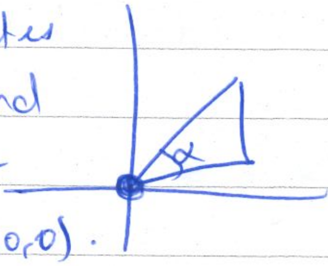
argument of constant (positive) = 0 as $\sec^2\alpha > 0$ for $\alpha \in \mathbb{R}$. $\therefore$ ~~...~~ $\therefore -\alpha + \alpha = 0$

5d) continued

e.g. $\arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \alpha$



$\therefore$ it is the same diagram as ~~[illegible]~~ z_3 just translates both z_1 and z_2 back to the origin $(0,0)$.



$\arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = -\alpha$

because ~~it calculates~~ $\tan(\pi - \alpha) = -\tan\alpha$

2e) $f(x) = (\log_2 x)^2 + 6m(\log_2 x) + n$

$m, n \in \mathbb{R}$.

$u = \log_2 x$

$du = \frac{1}{x\ln 2}$

$\therefore$ ~~8~~ $\left[\log_2 \frac{1}{8}\right]^2 + 6m\left(\log_2 \frac{1}{8}\right) + n = -2$

~~$\therefore$ 9 $-18m + n = -11$~~

~~$f'(x) = 2 \cdot \log_2 \frac{1}{8} \cdot \frac{1}{x\ln 2} + \frac{6m}{x\ln 2}$~~

~~$\therefore 2 \cdot -3 \cdot \frac{1}{-3\ln 2} + \frac{6m}{-3\ln 2} = 0$~~

~~$\therefore \frac{2}{\ln 2} - \frac{2m}{\ln 2} = 0 \Rightarrow m = 0.$~~

~~$\therefore n = -11$~~

$(-3)^2 + 6m(-3) + n = -2$

$\Rightarrow -18m + n = -11$

2e continued). $f'(x) = \frac{2 \circ \log_2 x}{x \ln 2} + \frac{6m}{x \ln 2}$

$\therefore \frac{2(\cancel{-3})}{(\cancel{-3})\ln 2} + \frac{-2 \; 6m}{(\cancel{-3})\ln 2} = 0$

$m = 1 \quad n = 7$

$\therefore (m, n) = (1, 7)$.

3a) $\cos\theta + \sin\theta = \frac{\sin\theta}{1 - \cot\theta} + \frac{\cos\theta}{1 - \tan\theta}$

$\therefore$ LHS $= \frac{\cos\theta \cdot (1 - \tan\theta)}{(1 - \tan\theta)} + \frac{\sin\theta \cdot (1 - \cot\theta)}{1 - \cot\theta}$

$= \frac{\cos\theta - \sin\theta}{1 - \tan\theta} + \frac{\sin\theta - \cos\theta}{1 - \cot\theta}$

$= \left[\frac{\sin\theta}{1 - \cot\theta} + \frac{\cos\theta}{1 - \tan\theta}\right] - \left[\frac{\sin\theta}{1 - \tan\theta} + \frac{\cos\theta}{1 - \cot\theta}\right]$

$=$ RHS $- \left[\frac{\cancel{\sin\theta} - \cancel{\cos\theta} + \cancel{\cos\theta} - \cancel{\sin\theta}}{(1 - \tan\theta)(1 - \cot\theta)}\right]$

$=$ RHS $- 0$

$=$ RHS as required.

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93202A

Annotated Outstanding Scholarship Exemplar Template

Subject	Scholarship Calculus	*Standard*	93202	*Total score*	36

Q	*Grade score*	*Annotation*
1	**8**	The candidate demonstrated accuracy in applying algebra and calculus skills by noting the domains of functions in 1a, 1b. The argument in 1c and 1e were logically set out with clarity.
2	**8**	The candidate displayed elegance in their dealing with the logarithm functions in 2a and 2e. They showed ability in applying the indirect 'hint' provided in finding the derivatives of a complicated composite function in 2d.
3	**8**	The candidate showed competence in manipulating trig identities in 3a. They also exhibited aptitude in applying permutation principles in 3d.
4	**8**	The candidate yet again showed strong algebra skills in manipulating complicated expressions in 4b, and completed it with flair and elegance. They also displayed ability in finding patterns in 4c.
5	**4**	The candidate successfully transformed a coordinate geometry problem into an algebraic equation solving question in 5a. In 5b, they recognised the square of a real number can't be negative, however they pre-maturely cancelled 'y' before realising it might be 0.