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SUPERVISOR'S USE ONLY

TOP SCHOLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
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Tick this box if you have NOT written in this booklet

Scholarship 2021
Calculus

Time allowed: Three hours
Total score: 40

ANSWER BOOKLET

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write your answers in this booklet.

Make sure that you have Formulae Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

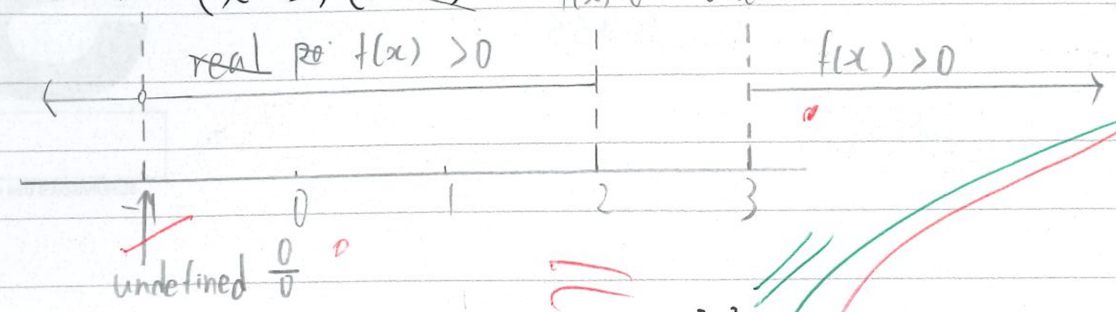
Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	

ASSESSOR'S USE ONLY

1a. $f(x) = \dfrac{(x-2)\cancel{(x+1)}}{(x-3)\cancel{(x+1)}}$



1b. let $y = \sqrt{x}$. $\quad (y^2)^{y^2 \cdot y} = (y^2)^{2y^2}$

$$y^{2y^3} = y^{4y^2}$$

$$y^{2y^3} - y^{4y^2} = 0$$

$$(y^{2y^3 - 4y^2} - 1)\, y^{4y^2} = 0$$

$y^{4y^2} = 0$ or $y^{2y^3 - 4y^2} = 1$

$2y^3 - 4y^2 = 0$ or $y = 1$

$y = 0$ or $(y-2)y^2 = 0$

$y = 2$ or $y = 0$

case 1: $y = 0$ then $x = 0^2 = 0$

case 2: $y = 1$ then $x = 1^2 = 1$

case 3: $y = 2$ then $x = 2^2 = 4$

alternative method for checking: $x^{x^{\frac{3}{2}}} - x^{2x} = 0$

$$(x^{(x^{\frac{3}{2}} - 2x)} - 1)\, x = 0$$

$x = 0$ or $x^{(x^{\frac{3}{2}} - 2x)} = 1$

$x = 1$ or $x^{\frac{3}{2}} - 2x = 0 \Rightarrow \sqrt{x} = 2,\ x = 4$

ASSESSOR'S USE ONLY

1c. let that vertical line be $x = a$

at the endpoints $2x^2 - x - 1 = -2x^2 - x + 1$

$4x^2 - 2 = 0$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$

$$\int_{-\sqrt{\frac{1}{2}}}^{a} (2x^2 - x - 1) - (-2x^2 - x + 1)\, dx = \int_{a}^{\sqrt{\frac{1}{2}}} (2x^2 - x - 1) - (-2x^2 - x + 1)\, dx$$

$$\int_{-\sqrt{\frac{1}{2}}}^{a} (4x^2 - 2)\, dx = \int_{a}^{\sqrt{\frac{1}{2}}} (4x^2 - 2)\, dx$$

$$\left[\frac{4x^3}{3} - 2x\right]_{-\sqrt{\frac{1}{2}}}^{a} = \left[\frac{4x^3}{3} - 2x\right]_{a}^{\sqrt{\frac{1}{2}}}$$

$$\left(\frac{4a^3}{3} - 2a\right) - \left(\frac{4\times\left(-\sqrt{\frac{1}{2}}\right)^3}{3} - 2\left(-\sqrt{\frac{1}{2}}\right)\right) = \left(\frac{4\left(\sqrt{\frac{1}{2}}\right)^3}{3} - 2\left(\sqrt{\frac{1}{2}}\right)\right) - \left(\frac{4a^3}{3} - 2a\right)$$

$$2\left(\frac{4a^3}{3} - 2a\right) = \cancel{(}\frac{4\left(-\sqrt{\frac{1}{2}}\right)^3}{3} - 2\left(-\sqrt{\frac{1}{2}}\right)\cancel{)} + \cancel{(}\frac{4\left(\sqrt{\frac{1}{2}}\right)^3}{3} - \cancel{2(\sqrt{}}\ 2\left(\sqrt{\frac{1}{2}}\right) = 0$$

$\frac{4a^3}{3} - 2a = 0$

$\cancel{4a^3 - 6}\ 2a^3 - 3a = 0$

$(2a^2 - 3)a = 0$, $a = 0$ or $2a^2 - 3 = 0$

$a^2 = \frac{3}{2}$

$a = \pm\sqrt{\frac{3}{2}}$

$\because -\sqrt{\frac{1}{2}} < a < \sqrt{\frac{1}{2}}$ $\therefore$ reject $a = \pm\sqrt{\frac{3}{2}}$

$\therefore$ $a = 0$ only

1d. substitute $u = \sqrt{x+1}$ then $\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$, $dx = 2\sqrt{x+1}\,du$
$= 2u\,du$, endpoints $\sqrt{2+1} = \sqrt{3}$, $\sqrt{0+1} = 1$

$$= \int_1^{\sqrt{3}} \frac{u^2-1}{u} \cdot 2u\,du = \int_1^{\sqrt{3}} (2u^2-2)\,du$$

$$= \left[\frac{2u^3}{3} - 2u\right]_1^{\sqrt{3}} = \left(\frac{2(\sqrt{3})^3}{3} - 2\sqrt{3}\right) - \left(\frac{2\times1^3}{3} - 2\times1\right)$$

$$= 2\sqrt{3} - 2\sqrt{3} - \frac{2}{3} + 2 = \frac{4}{3}$$

1e. $\sin x - \cos x = 0$
$\sin x = \cos x$

$\tan x = \frac{\sin x}{\cos x} = 1$

$x = \tan^{-1}(1) = \frac{\pi}{4}$ or $= \frac{\pi}{4} + \pi = \frac{5\pi}{4}$
$y > 0 \quad \forall \quad x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$
$y < 0 \quad \forall \quad x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx + \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx + \int_{\frac{5\pi}{4}}^{\frac{9\pi}{4}} (\cos x - \sin x)\,dx$$

$$= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + \left[\sin x + \cos x\right]_{\frac{5\pi}{4}}^{\frac{9\pi}{4}}$$

$$= \cancel{-\cos\frac{5}{}} \sqrt{2} - -\sqrt{2} + \sqrt{2} - -\sqrt{2} = 4\sqrt{2}$$

ASSESSOR'S USE ONLY

2a. $\log_{\frac{a}{b}}(b) = \dfrac{\ln b}{\ln \frac{a}{b}} = \dfrac{\ln b}{\ln a - \ln b} = 5 \quad (1)$

$\log_{\frac{a}{b}}(\sqrt[3]{b} \times \sqrt[4]{a}) = \dfrac{\ln(b^{\frac{1}{3}} \times a^{\frac{1}{4}})}{\ln \frac{a}{b}} = \dfrac{\frac{1}{3}\ln b + \frac{1}{4}\ln a}{\ln a - \ln b} \quad (2)$

$(1) \Rightarrow \ln b = 5\ln a - 5\ln b$

$5\ln b + \ln b = 6\ln b = 5\ln a$

$\ln a = \frac{6}{5}\ln b$

$\therefore (2) = \dfrac{\frac{1}{3}\ln b + \frac{1}{4} \times \frac{6}{5}\ln b}{\ln a - \ln b} = \dfrac{19}{30} \times \dfrac{\ln b}{\ln a - \ln b} = \dfrac{19}{30} \times 5 = \dfrac{19}{6}$

2b. ~~$g($ $x+y=11$~~ $\quad g(x,y) = x + y = 11$

~~maximise $f(x) = x^2 \cdot y^3$~~ maximise $f(x,y) = x^2 \cdot y^3$

$\nabla g(x,y) = \langle 1, 1 \rangle$

$\nabla f(x,y) = \langle 2xy^3, 3x^2y^2 \rangle$

$y = 11 - x, \quad f(x,y) = x^2(11-x)^3$

$\cancel{f'(x,y)} = \dfrac{df}{dx} = 2x(11-x)^3 + 3x^2(11-x)^2 \times -1 = 0$ at maximum

$2x(11-x)^3 = 3x^2(11-x)^2$

$x = 0$ or $11 - x = 0$ or $2(11-x) = 3x$

$x = 11 \qquad 22 - 2x = 3x$

$y = 11 - x = 0 \qquad 5x = 22$

$x = 4.4$

reject $\overbrace{x=0}^{\text{and } x=11}$ because specifies positive. at $x = 4.4$, $y = 11 - 4.4 = 6.6$

$4.4^2 \times 6.6^3 = 5565.92256$

$\frac{d^2f}{dx^2} = \cancel{2(11-x)^{-3}}$ ((CAS calculator) $\frac{d}{dx}\square, 2x(11-x)^3 -$

$3x^2(11-x)^2, \; x = 4.4 \quad = -958.32 < 0$

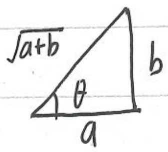
$\therefore x = 4.4$ is a maximum

$\because$ it is the only stationary point in the interval $x \in (0, 11)$

$\therefore$ it is the absolute maximum

2c. let $r\cos\theta = a$ and $r\sin\theta = b$

$r^2(\cos^2\theta + \sin^2\theta) = r^2 = a^2 + b^2$



$r = \sqrt{a+b}$

$\cos\theta = \frac{a}{\sqrt{a+b}}, \quad \theta = \cos^{-1}\left(\frac{a}{\sqrt{a+b}}\right)$

$f(x) = r\cos\theta\sin(\pi x + \alpha) + \cancel{b\cos(\pi x + \alpha)}\; r\sin\theta\cos(\pi x + \alpha) + 1$

$= r\sin(\pi x + \alpha + \theta) + 1$

$f(2020) = r\sin(2020\pi + \alpha + \theta) + 1 = 10$

$= r\sin(\alpha + \theta) + 1 = 10$

$r\sin(\alpha + \theta) = 9$

$f(2021) = r\sin(2021\pi + \alpha + \theta) + 1 = r\sin(\pi + \alpha + \theta) + 1$

$= -r\sin(\alpha + \theta) + 1 = -9 + 1 = -8$

2d. $\ln f(x) = \sin x \ln(x^2+1)$

$\frac{f'(x)}{f(x)} = \cos x \ln(x^2+1) + \sin x \cdot \frac{2x}{x^2+1}$

$f'(x) = (x^2+1)^{\sin x}\left[\cos x \ln(x^2+1) + \sin x \cdot \frac{2x}{x^2+1}\right]$

$f'(\frac{\pi}{2}) =$

$= \pi$

ASSESSOR'S USE ONLY

2e. $\cancel{f(x)=}$ $\frac{d}{dx}[\log_2 x] = \frac{d}{dx}\left[\frac{\ln x}{\ln 2}\right] = \frac{1}{x}\cdot\frac{1}{\ln 2} = \frac{1}{x\ln 2}$

$f'(x) = \frac{df}{d[\log_2 x]} \times \frac{d[\log_2 x]}{dx} = [2(\log_2 x) + 6m] \times \frac{1}{x\ln 2}$

at minimum point, $f'(\frac{1}{8}) = (2(\log_2 \frac{1}{8}) + 6m) \times \frac{1}{\frac{1}{8}\ln 2} = 0$

$2(\log_2 \frac{1}{8}) + 6m = -6 + 6m = 0$

$6m = 6$

$m = 1$

also given $f(\frac{1}{8}) = (\log_2 \frac{1}{8})^2 + 6\times 1\times(\log_2 \frac{1}{8}) + n = -2$

$(-3)^2 + 6\times -3 + n = -2$

$9 - 18 + n = -2$

$n = 7$

3a. $ax^2+bx+c = (x-\sin\theta)(x-\cos\theta)$

$x^2+\frac{b}{a}x+c = x^2-\sin\theta x-\cos\theta x+\sin\theta\cos\theta$

$\frac{b}{a} = -\sin\theta-\cos\theta$ by coefficient matching

$-\frac{b}{a} = \sin\theta+\cos\theta \quad (1)$

$$\frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta} = \frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}}$$

$$= \cancel{\frac{\sin^2\theta}{1-\cos\theta}}+\cancel{\cos^2\theta} \quad \frac{\sin^2\theta}{\sin\theta-\cos\theta}+\frac{\cos^2\theta}{\cos\theta-\sin\theta}$$

$$= \frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta} = \frac{\cancel{(\sin\theta-\cos\theta)}(\sin\theta+\cos\theta)}{\cancel{\sin\theta-\cos\theta}} = (1)$$

3b. x is a double root.

$16x^2-9(mx+2\sqrt{21})^2 = 144$

$16x^2-9m^2x^2-36\sqrt{21}mx-756 = 144$

$(16-9m^2)x^2-(36\sqrt{21}m)x-900 = 0$

$\cancel{x} = \quad \Delta = (-36\sqrt{21}m)^2-4\times-900\times(16-9m^2) = 0$

$27216m^2+57600-32400m^2 = 0$

$57600 = 5184m^2$

$m^2 = \frac{100}{9}$

$m = \pm\frac{10}{3}$

~~$(x--\sqrt{3})(x-\sqrt{3})(x-0) = ax^3-bx$~~

3c. $A(x--\sqrt{3})(x-\sqrt{3})(x-0) = ax^3-bx$ by factor theorem

$f(x) = A(x+\sqrt{3})(x-\sqrt{3})x = A(x^2-3)x = A(x^3-3x)$

$f'(x) = A(3x^2-3)$

$f'(0) = A(3\times 0^2 - 3) = -3A$

$f'(\sqrt{3}) = A(3\times(\sqrt{3})^2 - 3) = 6A$

$\tan 45° = |6A|$

~~3d~~ $\cancel{A} = |6A| = 1$

$A = \pm\frac{1}{6}$, but reject $A = -\frac{1}{6}$ because $A = a > 0$

$-3\times\frac{1}{6} = -\frac{1}{2}$

3di. $5! = 120$

~~3d~~

3dii. ~~treat~~ treat the 2 girls like one object

$6!\times 2 = 1440$

3diii. = all the ways − ways in which the 2 girls stand next to each other

$= 7! - 6!\times 2 = 3600$

ASSESSOR'S USE ONLY

4a. $\int \frac{1}{0.16A+D}\,dA = \int 1\,dt$

$\frac{\ln|0.16A+D|}{0.16} = t + c$

$D = 4500 + 500 = 5000$

~~A(to~~ $A(0) = 76000$

$\frac{\ln|0.16 \times 76000 + 5000|}{0.16} = 0 + c = c$

$c = \frac{\ln 17160}{0.16}$

$\frac{\ln|0.16A + 5000|}{0.16} = t + \frac{\ln 17160}{0.16}$

$\ln|0.16A + 5000| = 0.16t + \ln 17160$

$0.16A + 5000 = e^{0.16t + \ln 17160} = 17160e^{0.16t}$

$A = \frac{17160e^{0.16t} - 5000}{0.16} = 107250e^{0.16t} - 31250$

$A(10) = 107250e^{0.16 \times 10} - 31250 = \$499962.73 < \$500\,000$

$\therefore$ ~~not~~ won't be sufficient

4bi. $\int \frac{1}{y^3}\,dy = \int (x-1)\,dx$

$\frac{y^{-2}}{-2} = \frac{x^2}{2} - x + c$

$y^{-2} = -x^2 + 2x + c$

substitute $(0,a)$: $a^{-2} = -0^2 + 2\times 0 + c = c$

so $y^{-2} = -x^2 + 2x + a^{-2}$

$y = (-x^2 + 2x + a^{-2})^{-\frac{1}{2}}$

ASSESSOR'S USE ONLY

$$y = \frac{1}{\sqrt{-x^2+2x+\frac{1}{a^2}}}$$

4bii. $-x^2+2x+\frac{1}{a^2} > 0$ for ~~y to be defined~~ $y \in \mathbb{R}$

$$x^2 - 2x - \frac{1}{a^2} \cancel{>0} < 0$$

endpoints: $x = \frac{--2 \pm \sqrt{(-2)^2 - 4 \times 1 \times -\frac{1}{a^2}}}{2 \times 1}$

$$x = \frac{2 \pm \sqrt{4+\frac{4}{a^2}}}{2} = 1 \pm \sqrt{1+\frac{1}{a^2}}$$

~~4bii.~~

domain: $1-\sqrt{1+\frac{1}{a^2}} < x < 1+\sqrt{1+\frac{1}{a^2}}$

~~range:~~ $\because$ ~~$\sqrt{-x^2+2x+\frac{1}{a^2}} > 0$~~

~~$\therefore y > 0$~~

$$-x^2+2x+\frac{1}{a^2}$$

$$= -((x-1)^2 - 1 - \frac{1}{a^2}) \leqslant -(0-1-\frac{1}{a^2}) = 1 + \frac{1}{a^2}$$

range: $y = \frac{1}{\sqrt{-x^2+2x+\frac{1}{a^2}}} \geqslant \frac{1}{\sqrt{1+\frac{1}{a^2}}}$

$\because$ $-x^2+2x+\frac{1}{a^2}$ can approach 0 $\therefore$ there is no upper limit

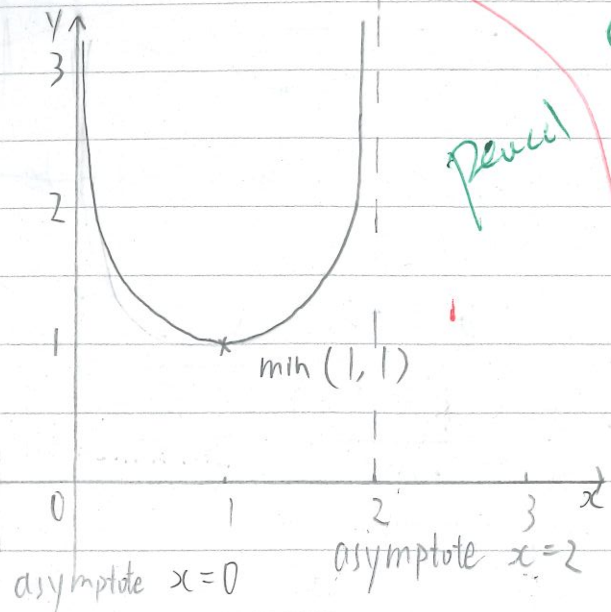
note: $x \in \mathbb{R}$, $y \in \mathbb{R}$

4biii: $\lim_{a\to+\infty}\left[\frac{1}{\sqrt{-x^2+2x+\frac{1}{a^2}}}\right] = \frac{1}{\sqrt{-x^2+2x}}$

domain: $1-\sqrt{1+0} < x < 1+\sqrt{1+0}$

note: $\lim_{a\to+\infty}\left[\frac{1}{a^2}\right] = 0$ $\qquad 0 < x < 2$

range: $y \geq \frac{1}{\sqrt{1+0}} = 1$



pencil

4c. $\cancel{T_1 = \sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} = 1.5 = \frac{3}{2}}$ $\quad \cancel{n(n+1)=2}$

$\cancel{= 1+1+\frac{1}{4}}$

$$T_n = \sqrt{\frac{n^2(n+1)^2+(n+1)^2+n^2}{n^2(n+1)^2}}$$

$$= \frac{\sqrt{(n^2+1)((n+1)^2+1)-1}}{n(n+1)}$$

$$= \frac{\sqrt{n^4+2n^3+n^2+n^2+2n+1+n^2}}{n(n+1)}$$

$$= \frac{\sqrt{n^4+2n^3+3n^2+2n+1}}{n(n+1)}$$

$$(n^2+n+1)^2 = \cancel{n^4}+\cancel{n^2}+1+\cancel{2n^3}+2n+\cancel{2n^2} = n^4+2n^3+3n^2+2n+1$$

$$T_n = \frac{n^2+n+1}{n(n+1)} = \frac{n^2+n+1}{n^2+n} = \frac{n^2+n}{n^2+n} + \frac{1}{n^2+n} = 1 + \frac{1}{n^2+n}$$

$$\sum_{n=1}^{2021} T_n = \sum_{n=1}^{2021} \left(\frac{1}{n^2+n}\right) + 2021$$

ASSESSOR'S USE ONLY

$\cancel{\#}$ $\quad \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4}$

$\frac{1}{1\times 2} - \frac{1}{2\times 3} + \frac{1}{2\times 3} -$

$\cancel{\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} = \frac{n+2-n}{n(n+1)(n+2)}}$

$\frac{1}{n(n+1)} - \frac{1}{n(n+2)} + \frac{1}{n(n+2)} + \frac{1}{(n+1)(n+2)}$

$\cancel{\frac{(n+2)-(n+1)}{n(n+1)(n+2)}}$

Let $\cancel{S_n =}$ $\quad S_k = \sum_{n=1}^{k} \left(\frac{1}{n(n+1)}\right)$

$\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} = \frac{\cancel{n+2-n}}{n(n+1)(n+2)}$

$\therefore \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

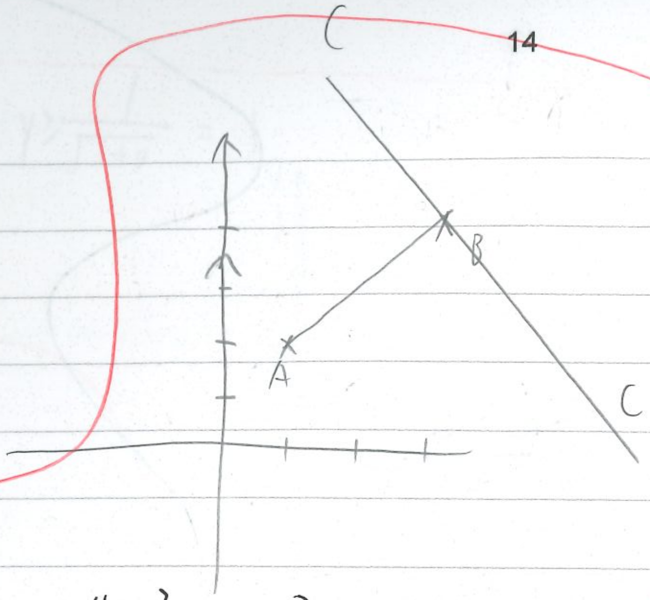
$\therefore \sum_{n=1}^{2021} \left(\frac{1}{n^2+n}\right) + 2021 = 2021 + \sum_{n=1}^{2021} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 2021 + \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{2021} - \frac{1}{2022}\right)$

$= 2021 + \left(1 - \frac{1}{2022}\right) = 2021\frac{2021}{2022}$

~~check~~ $\cancel{\frac{3}{2} = 1 + \left(1 - \frac{1}{2}\right), \quad \frac{3}{2} + \frac{7}{6} = 1 + 2 + \left(1 - \frac{1}{3}\right)}$

5a.



$$mAB = \frac{4-2}{3-1} = \frac{2}{2} = 1$$

$$mBC = -1 \div mAB = -1 \div 1 = -1$$

$$(y-4) = -(x-3)$$

$$y - 4 = -x + 3$$

$$y = 7 - x$$

$$|AB|^2 = (3-1)^2 + (4-2)^2 = 2^2 + 2^2 = 8$$

$$|BC|^2 = 4^2|AB|^2 = 16 \times 8 = 128$$

$$\begin{cases} (x-3)^2 + (y-4)^2 = 128 \\ y = 7 - x \end{cases}$$

$$(x-3)^2 + (7-x-4)^2 = 128$$

$$(x-3)^2 + (3-x)^2 = 128$$

$$2(\cancel{3} \rightarrow x-3)^2 = 128$$

$$(x-3)^2 = 64$$

$$x - 3 = \pm 8$$

$$x = 3 \pm 8 = 11 \text{ or } = -5$$

$$y = 7 - 11 = -4 \text{ or } y = 7 - -5 = 12$$

$C = (11, -4)$ ~~and~~ or $(-5, 12)$

5.b. $(x+iy)+\dfrac{1}{x-iy}=(x-iy)+\dfrac{1}{x+iy}$

$$(x+iy)+\frac{(x+iy)}{(x-iy)(x+iy)}=(x-iy)+\frac{(x-iy)}{(x+iy)(x-iy)}$$

$$(x+iy)+\frac{x+iy}{x^2+y^2}=(x-iy)+\frac{(x-iy)}{x^2+y^2}$$

$$(x+iy)(x^2+y^2)+(x+iy)=(x-iy)(x^2+y^2)+(x-iy)$$

$$(x+iy)(x^2+y^2+1)=(x-iy)(x^2+y^2+1)$$

either $x^2+y^2+1=0$ | or $x+iy=x-iy$

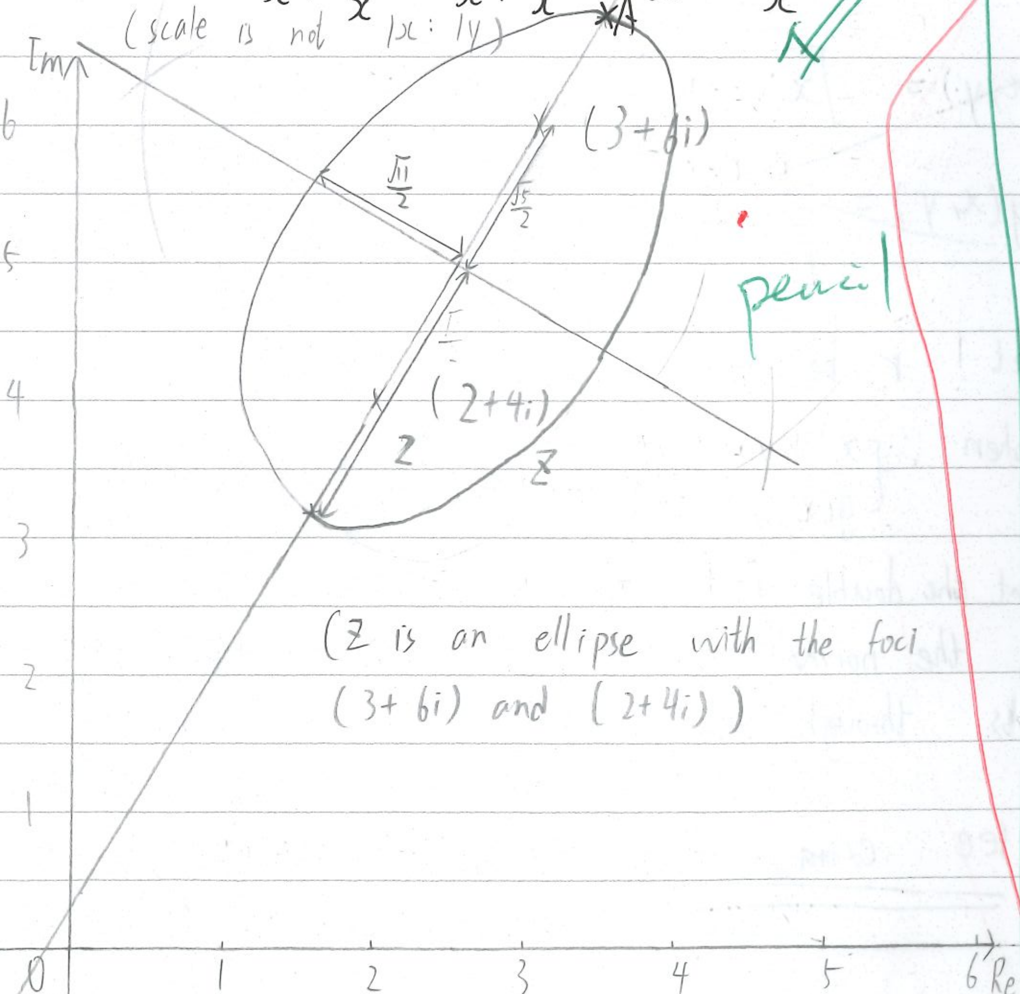
$\because x^2\geq 0,\ y^2\geq 0$ and $1>0$ | $2iy=0$

$\therefore$ this is impossible | $y=0$

$\therefore z=x$, where $x\in\mathbb{R}$, is the only class of solution

checking by substitution $x+\dfrac{1}{\bar{x}}=x+\dfrac{1}{x}=\bar{x}+\dfrac{1}{x}$

5ci.



$\left(\frac{\sqrt{5}}{2}\right)^2+x^2=2^2$

$(3-2)^2+(6-4)^2=5$

$x^2 + \frac{5}{4} = 4$

$x^2 = 4 - \frac{5}{4} = \frac{11}{4}$

$x = \frac{\sqrt{11}}{2}$

midpoint = center = $\frac{2+3}{2} + \frac{4i+6i}{2}$

$= 2.5 + 5i$

~~$z - 2.5$ let $z = x + iy$ then~~

let $z = x + iy$

$\cancel{\frac{(x-2.5)^2}{2^2}} +$

$$g(x,y) = |(x-2) + (y-4)i| + |(x-3) + (y-6)i| = 4$$

$$\cancel{g(x)} \quad \sqrt{(x-2)^2 + (y-4)^2} + \sqrt{(x-3)^2 + (y-6)^2} = 4$$

$$\cancel{f(x,y) = \sqrt{x^2 + y^2}}$$

$$\cancel{\nabla g(x,y) = \frac{2(x-2)}{\sqrt{\ }}}$$

Let r be the maximum of $|z|$

then $\begin{cases} x^2 + y^2 = r^2 \\ g(x,y) = 4 \end{cases}$ has a double root

at the double root, the two functions have the same gradient, and the normal of a point on $x^2+y^2=r^2$ always pass through the origin

see extra space (after 5d) Seen

ASSESSOR'S USE ONLY

5d. $|AC| \times |BA| =$

$\cancel{\frac{1}{2}(z}$ $\frac{1}{4}(z_2 - z_3)^2 (\tan^2\alpha + 1)$

$= \left(\frac{1}{2}(z_2 - z_3)\tan\alpha\right)^2 + \left(\frac{1}{2}(z_2 - z_3)\right)^2$

WLOG ~~let~~ $|AB| = |AC| = 1$ unit

Let θ be the angle AC form with the Re axis

$LHS = (z_2 - z_1)(z_1 - z_3)$

$= -\text{cis}\,\theta \times \text{cis}(2\alpha + \theta)$

$= -\text{cis}(2\alpha + 2\theta)$

$|BC|^2 = |AB|^2 + |BC|^2 - 2|AB||BC|\cos(\pi - 2\alpha)$

$= 2 \cancel{+1} - 2\cos(\pi - 2\alpha)$

$= 2 + 2\cos(2\alpha)$

$z_2 - z_3 = \sqrt{2 + 2\cos(2\alpha)} \times \text{cis}(\alpha + \theta)$

$RHS = \frac{1}{4}(2 + 2\cos(2\alpha))\,\text{cis}(2\alpha + 2\theta)\sec^2\alpha$

prove $\frac{1}{4}(2 + 2\cos(2\alpha))\sec^2\alpha = 1$

$\cancel{\frac{1}{4}}(2 + 2\cos(2\alpha)) = 4\cos^2\alpha$

$2 + 4\cos^2\alpha - 2 = 4\cos^2\alpha$ ✓ ???

okay try again:

WLOG, $|AB| = |AC| = 1$ unit

Let θ be the angle AC forms with the Re axis

$LHS = \cancel{-(z_2 - z_1)(z_3 - z_1)}\ (z_2 - z_1)(z_1 - z_3)$

$= \text{cis}(\theta + 2\alpha)\,\text{cis}(\theta)$

$= e^{i(\theta + 2\alpha)} \times e^{i\theta} = e^{i(2\theta + 2\alpha)} = \text{cis}(2\theta + 2\alpha)$

$\cancel{RHS} = |BC| = \sqrt{|AC|^2 + |AB|^2 - 2|AB|\,|AC|\cos\angle BAC}$

$= \sqrt{1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(\pi - 2\alpha)}$

$= \sqrt{2 - 2\cos(\pi - 2\alpha)}$

$= \sqrt{2 + 2\cos(2\alpha)}$

$(z_2 - z_3) = |BC| \operatorname{cis}(\alpha + \theta)$

$RHS = \left(\frac{1}{2}\sqrt{2 + 2\cos(2\alpha)} \operatorname{cis}\cancel{\cos}(\alpha+\theta) \sec\alpha\right)^2$

$= \frac{1}{4}(2 + 2\cos(2\alpha))\, \cancel{\cos^2(\alpha+\theta)\sec} \left(e^{i(\alpha+\theta)}\right)^2 \sec^2\alpha$

$= \frac{1}{4}(2 + 2\cos(2\alpha))\, e^{i(2\alpha + 2\theta)} \sec^2\alpha$

$= \frac{1}{4}(2 + 2\cos(2\alpha))\sec^2\alpha \times \operatorname{cis}(2\alpha + 2\theta)$

$= \frac{1}{4}(2 + 2(2\cos^2\alpha - 1))\sec^2\alpha \times \operatorname{cis}(2\alpha + 2\theta)$

$= \frac{1}{4}(\cancel{2-2} + 4\cos^2\alpha) \times \frac{1}{\cos^2\alpha} \times \operatorname{cis}(2\alpha + 2\theta)$

$= \operatorname{cis}(2\alpha + 2\theta) = LHS$ as required

QED //

Q5cii extra space

$$\nabla g(x,y) = \left\langle \frac{2(x-2)}{\sqrt{(x-2)^2+(y-4)^2}}, \frac{2(y-4)}{\sqrt{(x-2)^2+(y-4)^2}} \right\rangle$$

let $f(x,y) = \sqrt{x^2+y^2}$ then

$$\nabla f(x,y) = \left\langle \frac{2x}{\sqrt{x^2+y^2}}, \frac{2y}{\sqrt{x^2+y^2}} \right\rangle$$

by Lagrange multiplier, at maximum of $f(x,y)$,

$$\frac{2(x-2)}{\sqrt{(x-2)^2+(y-4)^2}} = \lambda \frac{2x}{\sqrt{x^2+y^2}}$$ okay this is too complicated hmmmmmm...

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{2(x-2) + 2(y-4)\cancel{\frac{dy}{dx}} - \frac{x}{y}}{2\sqrt{(x-2)^2+(y-4)^2}} + \frac{2(x-3) + 2(y-6)\cancel{\frac{dy}{dx}} - \frac{x}{y}}{\sqrt{(x-3)^2+(y-6)^2}} = 0$$

WAIT!!!

Note that ~~the~~ ~~pri~~ ~~principal axis~~,
~~(0,0)~~, O, ~~the origin~~ the principal axis passes through the origin

$$\sqrt{2.5^2+5^2} + 2 = \frac{4+5\sqrt{5}}{2} = |OA|$$

Imagine a circle with center (0,0). gradually increase its radius until the loci of z is internally tangent to the circle. The point of tangency ~~is the I~~ generates the maximal value of $|z|$, because the radius of the circle = $|z|$, and we can no longer increase the radius, otherwise ~~ther~~ $\nexists$ ~~no~~ intersection between the loci of $|z|$ and the circle.

So, I claim that A on the ~~diagram~~ ~~Arg~~ Argand diagram on page 15 is the point of tangency.

Proof: at point of tangency the two curves have overlapping normals and tangents.

any normal of the circle pass through its center (0, 0)

~~the~~ Since A is on the principle axis, ~~A~~ the normal to the loci of $|z|$ (an ellipse ~~!~~) ~~pass through~~ is the principal axis.

the principal axis has the equation

$$\cancel{(y-3)=} \quad (y-6) = \frac{6-4}{3-2}(x-3) = 2(x-3)$$

$$y-6 = 2x-6$$

$$y = 2x$$

so it ~~x~~ is ~~OA~~ congruent to OA. so the two normals overlap.

~~A is $\sqrt{4}$~~ Let $F_1 = (2+4i)$ and $F_2 = 3+6i$

$4 = \cancel{k}$ ~~then A~~ $|F_1A| + |F_2A| = 2|BA|$, where B is the midpoint of F_1 and F_2

$B = (2.5+5i)$

$|BA| = \frac{4}{2} = 2, \quad |OB| = \sqrt{2.5^2+5^2}$

$|OA| = |OB| + \cancel{|BA} \; |BA| = 2+\sqrt{2.5^2+5^2}$ = my answer

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