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Scholarship 2022
Calculus

Time allowed: Three hours
Total score: 40

ANSWER BOOKLET

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write your answers in this booklet.

Make sure that you have Formulae Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area may be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	

ASSESSOR'S USE ONLY

QUESTION ONE

1(a) $|z+a| = \sqrt{a}\,|z+1|$

let $z = x+iy$

$|x+iy+a| = \sqrt{a}\,|x+iy+1|$

$\sqrt{(x+a)^2+y^2} = \sqrt{a}\sqrt{(x+1)^2+y^2}$

$(x+a)^2+y^2 = a\left((x+1)^2+y^2\right)$

$x^2+2ax+a^2+y^2 = a(x^2+2x+1)+ay^2$

$x^2+\cancel{2ax}+a^2+y^2 = ax^2+\cancel{2ax}+a+ay^2$

$x^2+y^2-a(x^2+y^2) = a-a^2$

$(x^2+y^2)(1-a) = a-a^2$

$x^2+y^2 = \dfrac{a-a^2}{1-a} = \dfrac{a\cancel{(1-a)}}{\cancel{1-a}} = a$

(~~we~~ we may discard $a=1$ as the question states that $a \neq 1$)

$\sqrt{x^2+y^2} = \sqrt{a}$

$|z| = \sqrt{a}$ (valid since a is positive)

1(b) $x+y = \frac{\pi}{4} \Rightarrow \tan(x+y) = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \dfrac{\tan x + \tan y}{1-\tan x\tan y} = 1$

since $\tan x + \tan y = 1$

$\Rightarrow \dfrac{1}{1-\tan x\tan y} = 1$

$\Rightarrow \cancel{1} - \tan x\tan y = \cancel{1}$

$\Rightarrow \tan x\tan y = 0$

so either $\tan x = 0$ or $\tan y = 0$.

CONTINUED

1(b) if $\tan x = 0$, then $x = 0, \pi, -\pi, 2\pi, -2\pi, \ldots$ etc.

i.e. in general $x = n\pi$, where $n \in \mathbb{Z}$

since $x + y = \frac{\pi}{4}$, $y = \frac{\pi}{4} - x$

$= \frac{\pi}{4} - n\pi.$

on the other hand, if $\tan y = 0$, then $y = n\pi$, $x = \frac{\pi}{4} - n\pi$.

so the two solution sets are, where $n \in \mathbb{Z}$,

1. $(x, y) = (n\pi, \frac{\pi}{4} - n\pi)$
2. $(x, y) = (\frac{\pi}{4} - n\pi, n\pi)$

1(c) $x^4 + x^3 - 4x^2 + x + 1 = 0$

Divide both sides by x^2:

$x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0.$

let $u = x + \frac{1}{x}$.

$\Rightarrow u^2 = \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$

$\Rightarrow x^2 + \frac{1}{x^2} = u^2 - 2.$

$\rightarrow \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 0$

$(u^2 - 2) + (u) - 4 = 0$

$u^2 + u - 6 = 0$

$(u-2)(u+3) = 0 \Rightarrow u = 2$ OR $u = -3$ $\longrightarrow$ continued

1(c) It is given that $x < 0$, so with $u = 2$ OR $u = -3$, we must reject $u = 2$, ~~[scribbled out]~~ $\therefore u = -3$.

This is because, when x is negative, $x + \frac{1}{x}$ is also negative, and when x is positive, $x + \frac{1}{x}$ is also positive.

so $u = -3$, i.e. $x + \frac{1}{x} = -3$

$$\Rightarrow \left(x + \frac{1}{x}\right)^3 = (-3)^3$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3\cancel{(x)}\cancel{\left(\frac{1}{x}\right)}\left(x + \frac{1}{x}\right) = -27$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3(-3) = -27$$

$$\Rightarrow x^3 + \frac{1}{x^3} = -27 + 9 = -18.$$

$$\therefore x^3 + \frac{1}{x^3} = -18.$$

QUESTION TWO

2(a) $x^2-4x+10 = k(x+1)^2$

$x^2-4x+10 = k(x^2+2x+1)$

$x^2-4x+10 = kx^2+2kx+k$

$(k-1)x^2 + 2x(k+2) + k-10 = 0$

$\therefore k \neq 1$

$\frac{\Delta}{4} = \left(\frac{b}{2}\right)^2 - ac$

$= (k+2)^2 - (k-1)(k-10)$

$= \cancel{k^2}+4k+4-(\cancel{k^2}-11k+10)$

$= 15k - 6 > 0$ for two distinct real roots.

$\Rightarrow k > \frac{6}{15}$

i.e. $k > \frac{2}{5}$

The roots must be the same sign, which means that the product of roots is positive. Therefore, $\frac{k-10}{k-1} > 0$ (since $r_1 r_2 = \frac{c}{a}$)

$\Leftrightarrow (k-10)(k-1) > 0$ (multiplying by the square $(k-1)^2$ does not change the sign of the LHS)

$\Leftrightarrow k<1$ OR $k>10$.

Collating all of these inequalities gives

$\frac{2}{5}<k<1 \;\bigcup\; k>10$ as the solution inequality for k.

union

2(b) Using the sine rule:

$$\frac{\sin(2\alpha)}{16} = \frac{\sin(\alpha)}{12}$$

$$\Rightarrow \frac{2\sin\alpha\cos\alpha}{16} = \frac{\sin\alpha}{12}$$

$$24\sin\alpha\cos\alpha = 16\sin\alpha$$

$$3\sin\alpha\cos\alpha = 2\sin\alpha$$

~~[illegible]~~

reject $\sin\alpha = 0$ assuming ΔABC is a normal ~~triangles~~ triangle whose angles cannot be $0°$ or $180°$.

$$\Rightarrow 3\cos\alpha = 2$$

$$\Rightarrow \cos\alpha = \frac{2}{3}.$$

since $0 < \alpha + 2\alpha < 180°$

i.e. $0 < 3\alpha < 180°$

i.e. $0 < \alpha < 60°$,

~~[illegible]~~ we have $\sin\alpha = \sqrt{1-\cos^2\alpha}$

$$= \sqrt{1-\left(\frac{2}{3}\right)^2}$$

$$= \frac{\sqrt{5}}{3}.$$

$\angle BAC = \pi - \alpha - 2\alpha$ ... triangle interior $\angle$s Σ to π or $180°$

$$\sin(\angle BAC) = \sin(\pi - 3\alpha)$$

$$= \sin(3\alpha)$$

$\longrightarrow$ continued

expand $(\text{cis}\,\theta)^3$

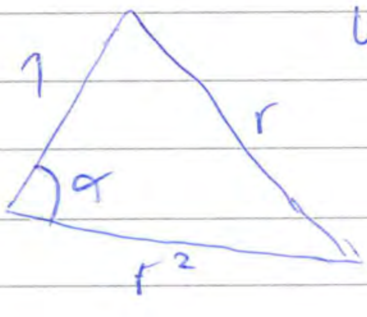
The triple angle formula
$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$
can be proven with De Moivre's theorem.

$$\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha$$
$$= 3\left(\frac{\sqrt{5}}{3}\right) - 4\left(\frac{\sqrt{5}}{3}\right)^3$$
$$= \frac{7\sqrt{5}}{27}.$$

Hence using $A = \frac{1}{2}bc\sin A$,

$$\text{Area} = \frac{1}{2}(16)(12)\left(\frac{7\sqrt{5}}{27}\right)$$
$$= \frac{224\sqrt{5}}{9}.$$

2(c) The sidelengths will be a, ar, ar^2.
But, due to similar triangles this can simplify to $1, r, r^2$.



Without loss of generality, let $r \geq 1$ (since if $r<1$, the order of the geometric sequence could be reversed to produce a geometric ratio greater than 1.)

Hence α is opposite the side r.
Using the cosine rule, $\cos\alpha = \dfrac{r^4 + 1 - r^2}{2(r^2)(1)}$.

$\longrightarrow$ continued

2(c) $\cos\alpha = \dfrac{r^4 - r^2 + 1}{2r^2}$

$$= \frac{r^4 + 2r^2 + 1 - 3r^2}{2r^2}$$

$$= \left(\frac{r^2+1}{\sqrt{2}\,r}\right)^2 - \frac{3}{2}$$

$$= \frac{1}{2}\left(r + \frac{1}{r}\right)^2 - \frac{3}{2}.$$

Due to the AM-GM inequality,

$$r + \frac{1}{r} \geq 2\sqrt{r \cdot \frac{1}{r}}$$
$= 2$, and this occurs when $r = 1$.

~~We~~ Given this information, $\left(r + \frac{1}{r}\right)^2 \geq 4$, and therefore $\cos\alpha \geq \frac{1}{2}(4) - \frac{3}{2} = \frac{1}{2}$.

We are trying to find the minimum α, so we want to maximise $\cos\alpha$, as shown by this graph.

cos α 

Hence for minimum α,

$$\cos\alpha = \frac{1}{2} \text{ i.e. } \alpha = 60^\circ.$$

This would make the triangle equilateral as we obtain equality when $r = 1$, i.e. sidelengths are 1, 1, 1. (equilateral)

QUESTION THREE

3(a) $f = \dfrac{e^{5x}\sqrt{x+1}}{e^{\sqrt{x+1}}}$

since $\log(ab) = \log a + \log b$.

$$\ln(f) = 5x - \sqrt{x+1} + \ln(\sqrt{x+1})$$
$$= 5x - \sqrt{x+1} + \frac{1}{2}\ln(x+1).$$ Differentiate:

$$\frac{f'(x)}{f(x)} = 5 - \frac{1}{2\sqrt{x+1}} + \frac{1}{2}\cdot\frac{1}{x+1}$$

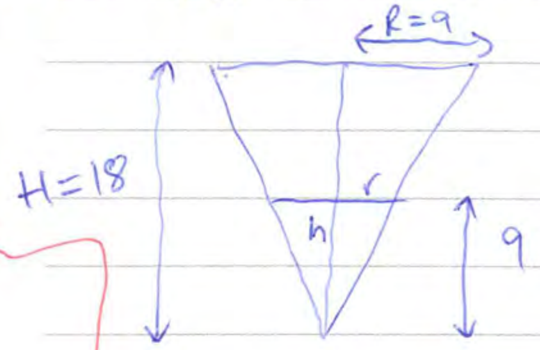
when $x=0$, $f(0) = \dfrac{\cancel{e^0}\,\sqrt{0+1}\cdot 1}{e^{\sqrt{0+1}}} = \dfrac{1}{e}.$

$$\therefore f'(0) = f(0)\left[5 - \frac{1}{2\sqrt{\cancel{0+1}}} + \frac{1}{2}\cdot\frac{1}{\cancel{0}+1}\right]$$

$$= \frac{1}{e}[5] = \frac{5}{e}.$$

3(b) For the conical dripper:

$$\frac{dh}{dr} = 2$$



$$V = \frac{1}{3}\pi r^2 h$$

note that by similar triangles,

$$h:r = H:r$$

$$\Rightarrow h = 2r \qquad (18 \div 9 = 2)$$

$$\text{so } V = \frac{2\pi}{3} r^3.$$

$$\frac{dV}{dr} = 2\pi r^2. \Rightarrow \frac{dr}{dV}_{r=4.5} = \frac{1}{2\pi(4.5)^2} = \frac{2}{81\pi}.$$

$$\frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dV} \cdot \frac{dV}{dt}$$

$$= 2 \cdot \frac{2}{81\pi} \cdot (-50)$$

... negative since out flow.

$$= -\frac{200}{81\pi}. \quad (\text{cm min}^{-1})$$

For the cylindrical flask:

surface area of coffe surface (circle):

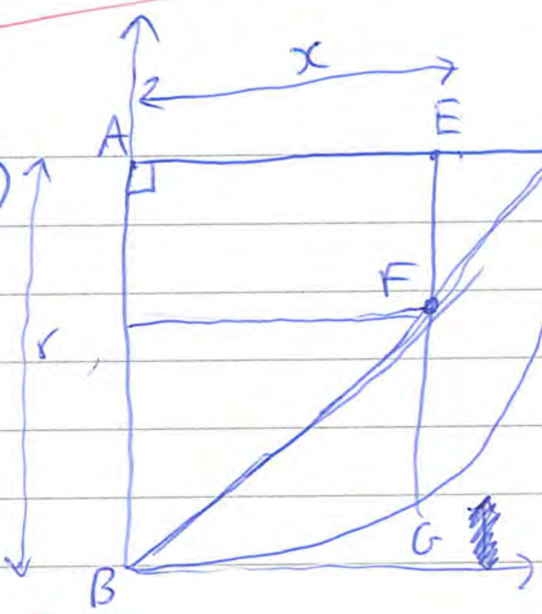
$$A = \pi r^2 = \pi (9)^2 = 81\pi.$$

$$\frac{dV}{dt} = A \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{1}{A}$$

$$= 50 \cdot \frac{1}{81\pi}.$$

Hence the ratio is $\dfrac{h'_{conical}}{h'_{cylindrical}} = \dfrac{-\frac{200}{81\pi}}{\frac{50}{81\pi}} = -4.$

3(c)



Need to maximise FG. Let A have coordinates $(0, r)$. Let B be the origin.

The cartesian equation of the arc BC is

$$x^2 + (y-r)^2 = r^2$$

i.e.

$$(y-r)^2 = r^2 - x^2$$

$$r - y = \sqrt{r^2 - x^2}$$

$$y = r - \sqrt{r^2 - x^2}.$$

Let the x-coordinate of E be x.

The equation of $\overline{BC}$ is $y = x$, so the point F is $F: (x, x)$

$$FG = EG - EF$$

$$= \left(\sqrt{r^2 - x^2}\right) - (r - x)$$

$$= \sqrt{r^2 - x^2} + x - r.$$

$$\frac{d(FG)}{dx} = 1 - \frac{x}{\sqrt{r^2 - x^2}} = 0$$

$$1 = \frac{x}{\sqrt{r^2 - x^2}}$$

$$r^2 - x^2 = x^2$$

$$r^2 = 2x^2 \rightarrow x = \frac{r}{\sqrt{2}}.$$

continued →

with $x = \frac{r}{\sqrt{2}}$, the ~~area~~ max length will be

$$FG_{max} = \sqrt{r^2 - x^2} + x - r$$

$$= \sqrt{r^2 - \frac{r^2}{2}} + \frac{r}{\sqrt{2}} - r$$

$$= \sqrt{\frac{r^2}{2}} + \frac{r}{\sqrt{2}} - r$$

$$= \sqrt{2}r - r$$

$$= r(\sqrt{2} - 1).$$

QUESTION FOUR

4(a) Pick the "or otherwise" option.

$$\cos^6\theta = (\cos^2\theta)^3$$

$$= \left(\frac{1+\cos 2\theta}{2}\right)^3$$

$$= \frac{(1+\cos 2\theta)^2(1+\cos 2\theta)}{8}$$

$$= \frac{(\cos^2 2\theta + 2\cos 2\theta + 1)(\cos 2\theta + 1)}{8}$$

$$= \left[\frac{1+\cos 4\theta}{2} + 2\cos 2\theta + 1\right]\left[\cos 2\theta + 1\right]\cdot\frac{1}{8}$$

$$= \frac{1}{16}\left[1+\cos 4\theta + 4\cos 2\theta + 2\right]\left[\cos 2\theta + 1\right]$$

$$= \frac{1}{16}\left[\cos 4\theta + 4\cos 2\theta + 3\right]\left[\cos 2\theta + 1\right]$$

$$= \frac{1}{16}\left[\cos 4\theta\cos 2\theta + 4\cos^2 2\theta + \overset{7}{\cancel{3}}\cos 2\theta + \cos 4\theta + \cancel{4\cos 2\theta} + 3\right]$$

using the identity: $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$= \frac{1}{32}\left[2\cos 4\theta\cos 2\theta + 8\cos^2 2\theta + 14\cos 2\theta + 2\cos 4\theta + 6\right]$$

$$= \frac{1}{32}\left[\cos 6\theta + \cos 2\theta + 4(1+\cos 4\theta) + 14\cos 2\theta + 2\cos 4\theta + 6\right]$$

$$= \frac{1}{32}\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right]$$

$$= \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16},$$ as required.

1.17809

4(b)

$$I = \int_0^1 y\,dx$$

bounds are "the wrong way" as we want to integrate from left to right.

substitute $x = \cos^3 t$,

$$dx = (3\cos^2 t)(-\sin t)\,dt$$

$$y = \sin^3 t$$

$$I = \int_{\frac{\pi}{2}}^{0} (\sin^3 t)(3\cos^2 t)(-\sin t)\,dt$$

$$= \int_{\pi/2}^{0} -3\sin^4 t\cos^2 t\,dt$$

note that $\sin^4\theta = (1-\cos^2\theta)^2$

$$= \cos^4\theta - 2\cos^2\theta + 1$$

and also that $\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2$

$$= \frac{1 + 2\cos 2\theta + \cos^2 2\theta}{4}$$

$$= \frac{1}{8}\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

$$I = \int_{\pi/2}^{0} -3(\cos^4 t - 2\cos^2 t + 1)(\cos^2 t)\,dt$$

$$= \int_{\pi/2}^{0} -3\left[\cos^6 t - 2\cos^4 t + \cos^2 t\right]dt$$

continued

using 4(a) result.

4(b)

$$= \int_{\pi/2}^{0} -3\left[\frac{1}{32}\left(\cos 6t + 6\cos 4t + 15\cos 2t + 10\right) + 6\cdot\frac{1}{8}\left(\cos 4t + 4\cos 2t + 3\right) - \frac{3}{2}\left(1+\cos 2t\right)\right] dt$$

$$= \frac{-3}{32}\int_{\pi/2}^{0} \cos 6t + 6\cos 4t + 15\cos 2t + 10 - 8\left(\cos 4t + 4\cos 2t + 3\right) + 16\left(1+\cos 2t\right) dt$$

$$= \frac{-3}{32}\int_{\pi/2}^{0} \cos 6t - 2\cos 4t - \cos 2t + 2 \; dt$$

$$= \frac{3}{32}\left[\frac{1}{6}\sin 6t - \frac{2}{4}\sin 4t - \frac{1}{2}\cos 2t + 2t\right]_{0}^{\pi/2}$$

$$= \frac{3}{32}\left[2\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{3\pi}{32}.$$

I

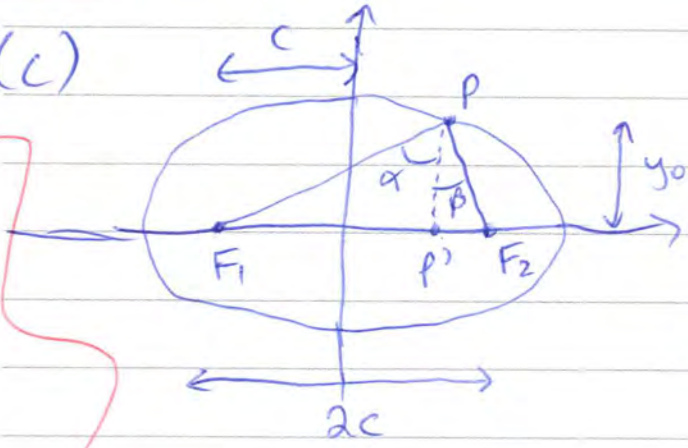
notice that I is exactly $\frac{1}{4}$ of the total area, so we finally have:

$$\text{Area}_{\text{astroid}} = 4 \times \frac{3\pi}{32}$$

$$= \frac{3\pi}{8} \approx 1.178$$

Area $\to \frac{1}{2}ab\sin C$
$\to \frac{1}{2}bh$

4(c)



Using the formula

$A = \frac{1}{2}bh$,

clearly $A = \frac{1}{2}(2c)(y_0)$
$= cy_0$.

Drop an altitude from P to P' on the major axis, so that F_1, P', F_2 collinear, and $\overline{F_1F_2} \perp PP'$.

Let $\angle F_1PP' = \alpha$
Let $\angle F_2PP' = \beta$.

$$\tan\theta = \tan(\alpha+\beta)$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{F_1P'}{PP'} + \frac{F_2P'}{PP'}}{1 - \left(\frac{F_1P'}{PP'}\right)\left(\frac{F_2P'}{PP'}\right)}$$

$$= \frac{\frac{c+x_0}{y_0} + \frac{c-x_0}{y_0}}{1 - \frac{(c+x_0)(c-x_0)}{y_0^2}}$$

continued

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow x^2=a^2\left(1-\frac{y^2}{b^2}\right)$$

$$= \frac{\frac{2c}{y_0}}{\frac{y_0^2-(c^2-x_0^2)}{y_0^2}} = \frac{2cy_0}{y_0^2+x_0^2-c^2}$$

note that $\tan\theta = \tan\left(2\frac{\theta}{2}\right)$

$$= \frac{\tan\frac{\theta}{2}+\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{2t}{1-t^2}$$

(let $t=\tan\frac{\theta}{2}$)

$$\therefore \frac{\cancel{2}cy_0}{y_0^2+x_0^2-c^2} = \frac{\cancel{2}t}{1-t^2}$$

$$cy_0(1-t^2) = t(y_0^2+x_0^2-c^2)$$

$$(cy_0)t^2 + t(y_0^2+x_0^2-c^2) - cy_0 = 0. \quad ☆$$

~~Verify that $t=cy_0$ is a solution to ☆:~~

~~$(cy_0)^2 + cy_0(y_0^2+x_0^2-c^2) - cy_0$~~

~~$= cy_0\left(y_0^2 + a^2 - \frac{y_0^2}{b^2} - c^2 + cy_0\right)$~~

~~$= cy_0\left(y_0^2\left(1-\frac{1}{b^2}\right) + b^2 - 1 + cy_0\right)$~~

$\longrightarrow$ next page

The fact that $A = cy_0 = b^2 \tan\frac{\theta}{2}$ is true if and only if $t = \frac{cy_0}{b^2}$.

$t = cy_0/b^2$

Verify ~~t_1~~ is a solution to equation ☆:

$$cy_0 t^2 + t(y_0^2 + x_0^2 - c^2) - cy_0$$

$$= \frac{(cy_0)^3}{b^4} + \frac{cy_0}{b^2}\left(y_0^2 + a^2 - \frac{a^2 y_0^2}{b^2} - c^2\right) - cy_0$$

$$= \frac{(cy_0)^3}{b^4} + \frac{cy_0}{b^2}\left(y_0^2\left(1 - \frac{a^2}{b^2}\right) + b^2\right) - cy_0$$

$$= \frac{(cy_0)^3}{b^4} + \frac{cy_0}{b^2}\left(\left(\frac{b^2 - a^2}{b^2}\right)y_0^2 + b^2\right) - cy_0$$

$$= \frac{(cy_0)^3}{b^4} + \frac{cy_0}{b^2}\left(\frac{-c^2}{b^2} y_0^3 + b^2\right) - cy_0$$

$$= \cancel{\frac{(cy_0)^3}{b^4}} - \cancel{\frac{(cy_0)^3}{b^4}} + \cancel{cy_0} - \cancel{cy_0} = 0,$$

so $t = \frac{cy_0}{b^2}$ is a solution to ☆.

Due to sums and products of roots, the ~~other root of ☆~~ ☆ product of roots $= \frac{-cy_0}{cy_0} = -1$, so the other root of ☆ is $t = -\frac{cy_0}{b^2}$ →

Note that $t = -\frac{cy_0}{b^2}$ arises

as an "~~false~~ extraneous solution"

due to the possibility of y_0 being negative,

so if we take $y_0 > 0$, then

we have $t = \frac{cy_0}{b^2}$

and therefore $cy_0 = tb^2$

$\boxed{\therefore \text{Area} = b^2 \tan\frac{\theta}{2}}$

QUESTION FIVE

5(a) $\tan(~~\cdots~~ 90^\circ - x) = \dfrac{\sin(90^\circ - x)}{\cos(90^\circ - x)}$

$= \dfrac{\cos x}{\sin x} = \cot x.$

$\tan x \cdot \cot x = \dfrac{\cos x}{\sin x} \times \dfrac{\sin x}{\cos x} = 1$

$\cancel{\dfrac{\cos^2 x + \sin^2 x}{\sin x \cos x}}$

Therefore $(\tan 1^\circ)(\tan 89^\circ) = 1.$

$(\tan 2^\circ)(\tan 88^\circ) = 1$, etc.

$\ldots$

$(\tan 44^\circ)(\tan 46^\circ) = 1.$

This leaves the remaining $\tan 45^\circ$ term in the middle.

$\therefore \int_0^a \tan 1^\circ \times \tan 2^\circ \ldots \times \tan 88^\circ \times \tan 89^\circ \, dx$

$= \int_0^a \tan 45^\circ \, dx$

$= \int_0^a 1 \, dx$

$= [x]_0^a = a$

5(b). Check by differentiation:

$$\frac{d}{dx}\left[\ln\left|\sqrt{1+x^2}+x\right| + c\right]$$

arbitrary constant of integration

$$= \frac{\frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}} + 1}{\sqrt{1+x^2}+x}$$

$$= \frac{\frac{1}{\sqrt{1+x^2}}\cancel{\left(x+\sqrt{1+x^2}\right)}}{1\cdot\cancel{\left(x+\sqrt{1+x^2}\right)}}$$

$$= \frac{1}{\sqrt{1+x^2}}, \text{ as required.}$$

(the fundamental thm. of calculus yields the result.)

5(c) ~~[illegible]~~ Using the suggested substitution:

$$\frac{dy}{dx} = A \rightarrow \frac{d^2y}{dx^2} = \frac{dA}{dx}.$$

substituting into the differential equation:

$$x\frac{dA}{dx} = \frac{v_1}{v_2}\sqrt{1+A^2}$$, let $k = \frac{v_1}{v_2}$ for simplicity

$$\Rightarrow \int \frac{dA}{\sqrt{1+A^2}} = \int k \frac{dx}{x}$$

$$\Rightarrow \ln\left|\sqrt{1+A^2} + A\right| = k\ln|x| + c.$$

$$\Rightarrow \left|\sqrt{1+A^2} + A\right| = c_2 |x|^k$$

case #1: $v_2 = v_1$. then $k=1$, so

$$\sqrt{1+A^2} + A = \pm c_2 x$$

$$\sqrt{1+A^2} = \pm c_2 x - A$$

$$1+\cancel{A^2} = c_2{}^2x^2 \mp 2Axc_2 + \cancel{A^2}$$

$$\Rightarrow \mp 2Axc_2 = 1 - c_2{}^2x^2.$$

we need to evaluate c_2. ~~[illegible]~~
when $x=1$, $A=0$, so $c_2 = \pm 1$.

$$\pm 2\frac{dy}{dx}x = 1-x^2$$

$$\pm\frac{dy}{dx} = \frac{1-x^2}{2x}$$

$$\pm\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x} - 1\right) \text{ M1}$$

$$\pm y = \frac{1}{2}\left(\ln|x| - x\right) + c_3,$$

but when $x = 1$, $y = 0$, so

$$0 = -\frac{1}{2} + c_3 \Rightarrow c_3 = \frac{1}{2}$$

$$\Rightarrow \pm y = \frac{1}{2}\left(\ln|x| - x + 1\right)$$

By the nature of the original differential equation, $\frac{d^2y}{dx^2} > 0$ when $x > 0$

so needs to be concave up. if $x > 0$.

so we can resolve ~~disregard~~ the abs value on $\ln|x|$ and $\pm$, picking the negative root for $x > 0$, ~~and the + root for $x < 0$.~~

~~This leads to the same thing, but reflected over the y axis.~~ ($x = 1$ exists $\Rightarrow$ pick $x > 0$)

$$\rightarrow \boxed{y = \frac{1}{2}\left(-\ln x + x - 1\right).} \text{ C}$$

case #2 : $V_2 \neq V_1$.

Return to ~~...~~

$$\left|\sqrt{1+A^2} + A\right| = c_2 |x|^k.$$

we can still evaluate c_2 as equal to 1, by substituting $(x, A) = (1, 0)$.

also we can take $x > 0$,

$$\therefore \sqrt{1+A^2} + A = x^k$$

$$\sqrt{1+A^2} = x^k - A$$

$$1 + \cancel{A^2} = x^{2k} - 2Ax^k + \cancel{A^2}$$

~~...~~

$$2A\,x^k = x^{2k} - 1$$

$$2\frac{dy}{dx} = x^k - x^{-k}.$$

note: this is a seperate case, since the absence of the possibility of $k=1$ means that there will not be a natural log integral (whereas $\int x^{-1}dx = \ln|x| + c \ldots$)

$$2y = \frac{x^{k+1}}{k+1} - \frac{x^{-k+1}}{-k+1} + 2c_4$$

$$y = \frac{1}{2}\left[\frac{x^{k+1}}{k+1} + \frac{x^{-k+1}}{k-1}\right] + c_4. \longrightarrow$$

when $x=1$, $y=0$, so

$$\frac{1}{2}\left[\frac{1}{k+1}+\frac{1}{k-1}\right]+c_4=0$$

$$\Rightarrow c_4=-\frac{1}{2}\left[\frac{1}{k+1}+\frac{1}{k-1}\right].$$

so this means that for case #2, the equation of the path is

$$\boxed{y=\frac{1}{2}\left[\frac{x^{k+1}}{k+1}+\frac{x^{1-k}}{k-1}\right]-\frac{1}{2}\left[\frac{1}{k+1}+\frac{1}{k-1}\right], \quad \text{where } k=\frac{v_1}{v_2}.}$$

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