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SCHOLARSHIP EXEMPLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Tick this box if you have NOT written in this booklet

Scholarship 2022
Calculus

Time allowed: Three hours
Total score: 40

ANSWER BOOKLET

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write your answers in this booklet.

Make sure that you have Formulae Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area may be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	

ASSESSOR'S USE ONLY

Q1 (a) Let $z=x+yi$

$\because |z+a|=\sqrt{a}\,|z+1|$

$\therefore |x+yi+a|=\sqrt{a}\,|x+yi+1|$

$\sqrt{(x+a)^2+y^2}=\sqrt{a}\,\sqrt{(x+1)^2+y^2}$

$(x+a)^2+y^2=a[(x+1)^2+y^2]$

$x^2+2ax+a^2+y^2=a(x^2+2x+1+y^2)$

$x^2+2ax+a^2+y^2=ax^2+2ax+a+ay^2$

$(1-a)x^2+(1-a)y^2+2ax+a^2=2ax+a$

$(1-a)x^2+(1-a)y^2=a-a^2=a(1-a)$

$(1-a)(x^2+y^2)=a(1-a)$ $\quad\because a\neq 1 \therefore 1-a\neq 0$

$x^2+y^2=a$

$\because |z|=|x+yi|=\sqrt{x^2+y^2}=\sqrt{a}$.

(b) $\because \underset{\text{From (A)}}{x+y=\frac{\pi}{4}}$ $\quad\therefore y=\frac{\pi}{4}-x$

From (B)

$\therefore \tan x+\tan y=1$

$\tan x+\tan(\frac{\pi}{4}-x)=1$

~~tan x + tan~~ $\because \tan(\frac{\pi}{4}-x)=\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}=\dfrac{1-\tan x}{1+\tan x}$

$\therefore \tan x+\dfrac{1-\tan x}{1+\tan x}=1$

$\dfrac{(1+\tan x)\tan x+1-\tan x}{1+\tan x}=1$

$\dfrac{\tan x+\tan^2 x+1-\tan x}{1+\tan x}=1$

$\tan^2 x+1=1+\tan x$

$\tan^2 x-\tan x=0$

$\tan x(\tan x-1)=0$

Example solutions:

① $\tan x=0$

(when $n=0$)

$x_1=0$

$\therefore y_1=\frac{\pi}{4}-0=\frac{\pi}{4}$

② $\tan x-1=0$

$\tan x=1$

(when $n=0$)

$x_1=\frac{\pi}{4}$

$y_2=\frac{\pi}{4}-\frac{\pi}{4}=0$

① $\because \tan x=0$

$\tan x=\tan 0$

$\therefore x_1=n\pi+0=n\pi$

$y_1=\frac{\pi}{4}-n\pi$

② $\because \tan x=1$

$\tan x=\tan\frac{\pi}{4}$

$\therefore x_2=n\pi+\frac{\pi}{4}$

$y_2=\frac{\pi}{4}-n\pi-\frac{\pi}{4}=-n\pi$

$\therefore$ For x and y to be real,

the solutions are $\begin{cases} x_1 = n\pi \\ y_1 = \frac{\pi}{4} - n\pi \end{cases}$ $\begin{cases} x_2 = n\pi + \frac{\pi}{4} \\ y_2 = -n\pi \end{cases}$

when n are any integers (eg. $n=0$, $n=1$, $n=2 \cdots$)

(c) $\because x^4 + x^3 - 4x^2 + x + 1 = 0$

Try $x=1$, $1^4 + 1^3 - 4\times 1^2 + 1 + 1 = 1+1-4+1+1 = 0$

$\therefore (x-1)$ is a factor

$\therefore (x-1)(x^3+ax^2+bx+c) = x^4 + ax^3 + bx^2 + cx - x^3 - ax^2 - bx - c$

$= x^4 + x^3 - 4x^2 + x + 1$

$\therefore x^4 + (a-1)x^3 + (b-a)x^2 + (c-b)x - c = x^4 + x^3 - 4x^2 + x + 1$

$\therefore a-1=1 \quad b-a=-4 \quad c-b=1$

$a=2 \quad b-2=-4 \quad c+2=1$

$b = 2-4 = -2 \quad c=-1$

$\therefore x^4 + x^3 - 4x^2 + x + 1 = (x-1)(x^3+2x^2-2x-1) = 0$

$\therefore x^3 + 2x^2 - 2x - 1 = 0$ will give the other roots when $x^4+x^3-4x^2+x+1=0$

$x^3+2x^2-2x = 1$

$(x^3+2x^2-2x) \div x^3 = \frac{1}{x^3}$

$1 + \frac{2}{x} - \frac{2}{x^2} = \frac{1}{x^3}$

$(x+\frac{1}{x})^2 = x^2 + 2\cdot x \cdot \frac{1}{x} + \frac{1}{x^2} = x^2 + \frac{1}{x^2} + 2$

$(x+\frac{1}{x})^3 = x^3 + 3x^2\cdot\frac{1}{x} + 3x\cdot\frac{1}{x^2} + \frac{1}{x^3} = x^3 + \frac{1}{x^3} + 3(x+\frac{1}{x})$

$\because$ when $x=1$, $1^3 + 2\times 1^2 - 2\times 1 - 1 = 1+2-2-1 = 0$

$\therefore x^4 + x^3 - 4x^2 + x + 1 = (x^2+3x+1)(x-1)^2 = 0$

($\because$ $x-1 \overline{)x^3+2x^2-2x-1}$ quotient x^2+3x+1:
$x^3 - x^2$
$3x^2-2x-1$
$3x^2-3x$
$x-1$
$x-1$
0)

$\therefore$ Let $x^2+3x+1 = 0$ will give the other two roots

$x^2+3x = -1$

$x^2+3x+\frac{9}{4} = \frac{9-4}{4} = \frac{5}{4}$

$(x+\frac{3}{2})^2 = \frac{5}{4}$

$x + \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$

$x_1 = \frac{\sqrt{5}-3}{2}$, $x_2 = \frac{-\sqrt{5}-3}{2}$

$\therefore$ All four solutions are:

$x_1 = x_2 = 1$, $x_3 = \frac{\sqrt{5}-3}{2}$, $x_4 = \frac{-\sqrt{5}-3}{2}$

$\because x<0$, $1>0$, $\frac{\sqrt{5}-3}{2} < 0$, $\frac{-\sqrt{5}-3}{2} < 0$.

$\therefore$ Both $x=\frac{\sqrt{5}-3}{2}$ or and $\frac{-\sqrt{5}-3}{2}$ satisfies the conditions

$\therefore$ ① $x=\frac{\sqrt{5}-3}{2}$, $\frac{1}{x}=\frac{2}{\sqrt{5}-3}$

$x^3+\frac{1}{x^3} \cancel{= (x+\frac{1}{x})^3-3(x+\frac{1}{x})}$

$\cancel{=(\frac{\sqrt{5}-3}{2}+\frac{2}{\sqrt{5}-3})^3-3(}$

$= (\frac{\sqrt{5}-3}{2})^3+(\frac{2}{\sqrt{5}-3})^3$

$=-18$

② $x=\frac{-\sqrt{5}-3}{2}$, $\frac{1}{x}=\frac{2}{-\sqrt{5}-3}$

$x^3+\frac{1}{x^3} = (\frac{-\sqrt{5}-3}{2})^3+(\frac{2}{-\sqrt{5}-3})^3$

$=-18$

$\therefore x^3+\frac{1}{x^3}=-18$ in both cases

Q2 (a) $x^2-4x+10=k(x+1)^2$

$x^2-4x+10 = k(x^2+2x+1)$

$x^2-4x+10 = kx^2+2kx+k$

$(1-k)x^2-(4+2k)x+(10-k)=0$

For the equation to have two distinct real roots:

$\Delta=b^2-4ac > 0$

$=[-(4+2k)]^2-4\times(1-k)\times(10-k)$

$=16+16k+4k^2-(4-4k)(10-k)$

$=16+16k+4k^2-(40-4k-40k+4k^2)$

$=16+16k+4k^2-40+44k-4k^2$

$=16+60k-40$

$=60k-24$

$60k-24>0$

$60k>24$

$k>\frac{2}{5}$

∵ The two distinct real roots have the same sign,
let one of the roots be h, then the other root must be:

Sum of Roots $= -\frac{b}{a} = \frac{4+2k}{1-k}$

The other root:

$$\frac{4+2k}{1-k} - h$$

Product of roots must be positive, as a negative number × a negative number = positive;
a positive number × a positive number = positive

$\therefore PoR = \frac{c}{a} = \frac{10-k}{1-k} > 0$

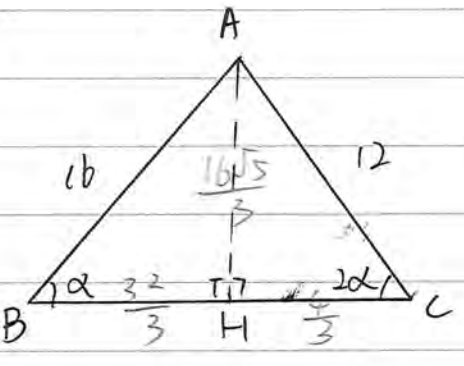
$k \neq 1$.

$$\therefore 10 - k > 0$$
$$-k > -10$$
$$k < 10$$

$\therefore \frac{2}{5} < k < 10$ and $k \neq 1$. Any k values within this region will give the equation 2 real distinct roots with the same sign.

Q2

(b)



Draw $AH \perp BC$

$\therefore \angle AHC = \angle AHB = 90°$

$\cos 2\alpha = \frac{HC}{12}$ $\therefore CH = 12\cos 2\alpha$

$\sin 2\alpha = \frac{AH}{12}$ $\therefore AH = 12\sin 2\alpha$

$\cos\alpha = \frac{BH}{16}$ $\therefore BH = 16\cos\alpha$

$\frac{\sin\alpha}{\cancel{\cos}} = \frac{AH}{16}$ $\therefore AH = 16\sin\alpha$

$\therefore AH = 12\sin 2\alpha = 16\sin\alpha$

$12 \times 2\sin\alpha\cos\alpha = 16\sin\alpha$

$24\sin\alpha\cos\alpha = 16\sin\alpha$

$\because \sin\alpha \neq 0$, $\alpha \neq 0$

$\therefore 24\cos\alpha = 16$

$\cos\alpha = \frac{16}{24} = \frac{2}{3}$

$\therefore BH = 16 \times \cos\alpha = 16 \times \frac{2}{3} = \frac{32}{3}$

$AH = \sqrt{16^2 - (\frac{32}{3})^2} = \frac{16\sqrt{5}}{3}$

$\cancel{CH = 12\cos 2\alpha = 12(2\cos^2\alpha - 1)}$

$\cancel{= 12 \times [2 \times (\frac{2}{3})^2 - 1]}$

$\cancel{= 12 \times (2 \times \frac{4}{9} - 1)}$

$\cancel{= 12 \times (1-}$

$CH = \sqrt{12^2 - (\frac{16\sqrt{5}}{3})^2}$

$= \sqrt{\frac{16}{9}} = \frac{4}{3}$

$\therefore Area = \frac{1}{2} \times BC \times AH$

(Next Page)

$\text{Area} = \frac{1}{2} \times BC \times AH$

$= \frac{1}{2} \times (BH + CH) \times AH$

$= \frac{1}{2} \times (\frac{32}{3} + \frac{4}{3}) \times \frac{16\sqrt{5}}{3}$

$= \frac{1}{2} \times \frac{36}{3} \times \frac{16\sqrt{5}}{3}$

$= \frac{1}{2} \times 12 \times \frac{16\sqrt{5}}{3}$

$= 6 \times \frac{16\sqrt{5}}{3}$

$= 2 \times 16\sqrt{5}$

$= 32\sqrt{5}$

Q2

(c) $\because \alpha$ is an angle inside a triangle

$\therefore \alpha > 0$

$(\cancel{a} > 0,\ b > 0,\ c > 0)$

Let the longest side of the triangle be c,
the shortest side of the triangle be b,
the side ~~o~~ opposite angle α be x.

$\therefore$ As the three side lengths form a geometric sequence with a common ratio of r,

$\therefore x = br$, $c = xr = br^2$

$\because c, b, x$ are three sides of a triangle

$\therefore c + b > x \qquad \therefore br^2 + b > br$ ①

$b + x > c \qquad br + b > br^2$ ②

$c + x > b \qquad br^2 + br > b$ ③

$r^2 + 1 > r$ ①

$r^2 - r + 1 > 0$

$r^2 - r + \frac{1}{4} > -1 + \frac{1}{4}$

$(r - \frac{1}{2})^2 > -\frac{3}{4}$

$\because (r - \frac{1}{2})^2 \geq 0$,

$-\frac{3}{4} < 0$

$\therefore$ Always true.
(for any value of r.)

$1 + r > r^2$ ②

$r^2 - r - 1 < 0$

$r^2 - r < 1$

$r^2 - r + \frac{1}{4} < \frac{5}{4}$

$(r - \frac{1}{2})^2 < \frac{5}{4}$

① $0 < r - \frac{1}{2} < \frac{\sqrt{5}}{2}$ or

② $r - \frac{1}{2} < 0$,

$r - \frac{1}{2} > -\frac{\sqrt{5}}{2}$

$\therefore$ ① $\frac{1}{2} < r < \frac{\sqrt{5}+1}{2}$ (1.618) or ② $\frac{1-\sqrt{5}}{2} < r < \frac{1}{2}$ (−0.618)

$r^2 + r > 1$ ③

$r^2 + r - 1 > 0$

$r^2 + r > 1$

$r^2 + r + \frac{1}{4} > \frac{5}{4}$

$(r + \frac{1}{2})^2 > \frac{5}{4}$

① $r + \frac{1}{2} > \frac{\sqrt{5}}{2}$, $r + \frac{1}{2} > 0$

$r > \frac{\sqrt{5}-1}{2}$, $r > -\frac{1}{2}$

$\therefore r > \frac{\sqrt{5}-1}{2}$ (0.618)

② $r + \frac{1}{2} < 0$, $r + \frac{1}{2} < \frac{-\sqrt{5}}{2}$

$r < -\frac{1}{2}$, $r < \frac{-\sqrt{5}-1}{2}$ (−1.618)

$r < -\frac{\sqrt{5}-1}{2}$

To satisfy ② and ③ at the same time.

$\therefore \frac{\sqrt{5}-1}{2} < r < \frac{\sqrt{5}+1}{2}$

$x^2 = b^2 + c^2 - 2bc\cos\alpha$ according to cosine rule

$br = b^2 + b^2r^4 - 2b \cdot br^2\cos\alpha$

$br = b^2 + b^2r^4 - 2b^2r^2\cos\alpha \quad \because b \neq 0$

$\therefore r = b + br^4 - 2br^2\cos\alpha$

$r = b(1 + r^4 - 2r^2\cos\alpha)$

$\frac{r}{b} = 1 + r^4 - 2r^2\cos\alpha$

$-2r^2\cos\alpha = \frac{r}{b} - 1 - r^4$

Q3 (a) Let $a = \sqrt{x+1}$ $\quad \frac{da}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}}$ $\therefore da = \frac{1}{2\sqrt{x+1}}dx$

$\therefore f(x) = \frac{e^{5x} \cdot a}{e^a}$

$f(x) = \frac{e^{5x}\sqrt{x+1}}{e^{\sqrt{x+1}}}$

$\ln f(x) = \log_e\left(\frac{e^{5x}\sqrt{x+1}}{e^{\sqrt{x+1}}}\right) = \log_e(e^{5x}\sqrt{x+1}) - \log_e e^{\sqrt{x+1}}$

$\ln f(x) = \ln(e^{5x}\sqrt{x+1}) - \sqrt{x+1}$

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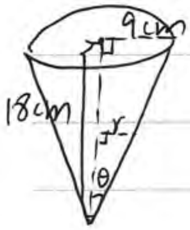
Q3 (b) $\frac{dV}{dt} = 50 cm^3$

Let the depth of coffee in the beaker be h cm, in the dripper be ~~d~~ y cm.



$18 \div 2 = 9cm$

Angle θ as shown on the left of the conical dripper stays the same

$\therefore$ Let the radius of the coffee at any instant be r cm as shown.

$\because$ Volume of cone $= \frac{1}{3}\pi r^2 \cdot y$

$V = \frac{1}{3}\pi r^2 \cdot y$

$\tan\theta = \frac{r}{y} = \frac{9}{18}$

$18r = 9y$

$2r = y$

$r = \frac{y}{2}$ $\quad y = 2r$

~~$\because \frac{dV}{dt} = 50 cm^3/min$~~

$\therefore$

$\frac{dV}{dt} = 50 cm^3/min$

$\therefore V = \frac{1}{3}\pi r^2 \cdot 2r$

$V = \frac{2}{3}\pi r^3$

$\therefore \frac{dV}{dr} = 2\pi r^2 \cancel{= 2\pi}$

~~$\frac{dy}{dr} = 2$~~

$\therefore V = \frac{1}{3}\pi r^2 \cdot y$

$= \frac{1}{3}\pi \frac{y^2}{4} \cdot y$

$= \frac{\pi}{12} y^3$

~~$\therefore \frac{dV}{dy} = \frac{dV}{dr} \times \frac{dr}{dy}$~~

~~$= 2\pi r^2 \cdot \frac{1}{2}$~~

~~$= \pi r^2$~~

~~$\therefore \frac{dV}{dy} = 2\pi \frac{y^2}{4} = \frac{\pi}{2} y^2$~~

$\frac{dV}{dy} = \frac{\pi}{4} y^2$

$\therefore \frac{dy}{dt} = \frac{dV}{dy} \times \frac{dt}{dV}$

$= \frac{\pi}{4} y^2 \times \frac{1}{50}$

$= \frac{\pi y^2}{4 \times 50} = \frac{\pi}{200} y^2$

volume of cylinder: $V_2 = \pi r^2 h$

$\because$ The radius of the cylinder is $18 \div 2 = 9cm$

$\therefore V_2 = \pi \times 9^2 h = 81\pi h$

$\therefore \frac{dV}{dh} = 81\pi$

$\therefore \frac{dh}{dt} = \cancel{\frac{dV}{dh} \times}$ $\frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{81\pi} \times 50 = \frac{50}{81}\pi$ of coffee mei

The $\frac{dV}{dt}$ (Rate of change of volume) in both the dripper and the cylinder are the same, because while coffee drips out ~~ou~~ of the dripper, the volume of coffee in the cone decreases at this rate; however, the coffee drips into the beaker, so the volume of the coffee in the beaker also increases at this rate.

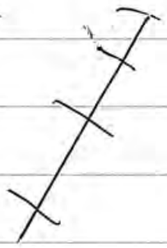
$\because \frac{dy}{dt} = \frac{\pi}{200} y^2$, $\frac{dh}{dt} = \frac{50\pi}{81}$

$\therefore$ When $y = 9cm$, $\frac{dy}{dt} = \frac{\pi}{200} \times 9^2 = \frac{81}{200}\pi$

$\therefore$ The ratio between the rate of change in depth ($\frac{dh}{dt}$) of the coffee in the beaker and the rate of change of depth of the coffee in the dripper ($\frac{dy}{dt}$) when depth in dripper is 9cm ($y = 9cm$) is:

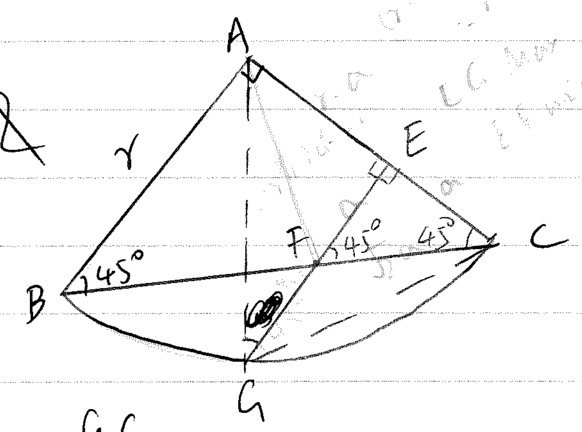
$\frac{dh}{dt} : \frac{dy}{dt} = \frac{50\pi}{81} : \frac{81}{200}\pi = \frac{50}{81} : \frac{81}{200} =$

~~(c)~~



Q3 (c) (continued on the next page)

Q3 (c)



Connect AG, GC.

$\because$ A is the center of the circle

$\therefore AC = AB = r = AG$

~~$\therefore \angle ABC = \angle AGE$~~ $\angle ABL = \angle AGC = 45^\circ$

$\because AB = AC = r,\ \angle BAC = 90^\circ$

$\therefore \angle ABC = \angle ACB = 45^\circ$

$\therefore \angle ABC = \cancel{\angle AGE}\ \angle ACG = 45^\circ$ $\quad \because \angle BAC = 90^\circ,\ GE \parallel AB \therefore \angle CEG = 90^\circ = \angle GEA$

$\because AG = AB = AC = r,\ \cancel{\angle AGE = 45^\circ}\ \angle ACG = 45^\circ,\ \cancel{GE}\ \angle CEG = 90^\circ = \angle GEA$

~~$\therefore AE = GE = \frac{r}{\sqrt{2}}$~~ $\quad EC = CF$ (Because $\angle ECF = \angle EFC = 45^\circ$)

~~$\therefore CE = AC - AE = r - \frac{r}{\sqrt{2}}$~~ $\quad \tan\angle ACB = \tan 45^\circ = 1$

~~$\because \angle ACB = 45^\circ,\ \angle CEG = 90^\circ$~~ $\quad \therefore \frac{EF}{EC} = \frac{AB}{AC} = 1$

~~$\therefore \angle EFC = \angle ACB = 45^\circ,\ EF = EC = r - \frac{r}{\sqrt{2}}$~~

~~$\because GE = \frac{r}{\sqrt{2}},\ EF = r - \frac{r}{\sqrt{2}}$~~ $\quad AE = r - EC = r - EF$

~~$\therefore FG = \frac{r}{\sqrt{2}} - r + \frac{r}{\sqrt{2}}$~~ $\quad$ ~~$AE \cdot \sqrt{2} = r$~~

~~$= \frac{2r - \sqrt{2}r}{\sqrt{2}}$~~ $\quad$ ~~$AE =$~~

$\quad GE = \sqrt{r^2 - (r - EC)^2}$

~~$= \sqrt{2}r - r = (\sqrt{2} - 1)r$~~ $\quad = \sqrt{r^2 - r^2 + 2rEC - EC^2}$

$\quad = \sqrt{2rEC - EC^2}$

~~$\frac{d(FG)}{dr} = \sqrt{2} - 1$~~

Let $EF = EC = a$, $AE = r - a$

$\therefore EG = \sqrt{r^2 - (r-a)^2}$

$= \sqrt{2ra - a^2}$

$FG = EG - FE = \sqrt{2ra - a^2} - a$

$FG^2 = (EG-FE)^2 = (\sqrt{2ra-a^2} - a)^2 = 2ra \cancel{-a^2} - 2\sqrt{2ra-a^2}\,a \cancel{+a^2}$

$FG^2 = 2ra - 2\sqrt{2ra-a^2}\,a$

FG is at maximum when FG^2 is at maximum

Q4 (a) $RHS = \frac{1}{32}\cos(4\theta+2\theta) + \frac{3}{16}\cos(2\theta+2\theta) + \frac{15}{32}\cos 2\theta + \frac{5}{16}$

$= \frac{1}{32}[\cos 4\theta \cos 2\theta - \sin 4\theta \sin 2\theta] + \frac{3}{16}[\cos 2\theta \cos 2\theta - \sin 2\theta \sin 2\theta]$

$+ \frac{15}{32}\cos 2\theta + \frac{5}{16}$

$=$

let $z = \operatorname{cis}\theta = \cos\theta + \sin\theta\, i = \frac{e^{i\theta}+e^{-i\theta}}{2} + \frac{e^{i\theta}-e^{-i\theta}}{2}$

$z^6 = \operatorname{cis}(6\theta) = \cos(6\theta) + \sin(6\theta) i$

$= \left(\frac{e^{i\theta}+e^{-i\theta}+e^{i\theta}-e^{-i\theta}}{2}\right)^6 = \left(\frac{2e^{i\theta}}{2}\right)^6 = (e^{i\theta})^6 = e^{6\theta i}$

$\cancel{z^6 = [\cos\theta + i\sin\theta]^6 = \cos^6\theta +}$

$z^6 = \frac{e^{6i\theta}+e^{-i\theta 6}}{2} + \frac{e^{i6\theta}-e^{-i\theta 6}}{2} = \frac{2e^{i6\theta}}{2} = e^{6\theta i}$

$\cos(6\theta) + \sin(6\theta) i =$

(Continued on next page)

Q4

(b) $x=\cos^3 t$

$y=\sin^3 t$

$0 \le t \le 2\pi$

$\frac{dx}{dt} = 3\cos^2 t \cdot (-\sin t) = -3\cos^2 t \sin t$

$\frac{dy}{dt} = 3\sin^2 t \cdot \cos t \quad = 3\sin^2 t \cos t$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 3\sin^2 t\cos t \times \frac{1}{-3\cos^2 t\sin t} = \frac{\sin t}{\cos t} = \tan t$

$\cancel{\int_0^1 y\,dx = \int}$

$\therefore y = \int \frac{dy}{dx}\,dx = \int \frac{\sin t}{\cos t} \cancel{dx}\, dt \cdot \frac{dx}{dt}$

$= \int \frac{\sin t}{\cancel{\cos t}} \times (-3\cos^2 t\sin t)\,dx$

$= \int -3\sin^2 t\cos t\,dx$

$= \int -3(1-\cos^2 t)\cos t\,dx$

$= \int -3 (\cos t - \cos^3 t)\,dx$

$= \int -3\cos t + 3\cos^3 t\,dx$

$= \int -3\sqrt[3]{x} + 3x\,dx$

$= \int -3x^{\frac{1}{3}} + 3x\,dx$

$= -3\int x^{\frac{1}{3}} - x\,dx$

$= -3\left[\frac{3}{4}x^{\frac{4}{3}} - \frac{x^2}{2}\right] + c$

$= -\frac{9}{4}x^{\frac{4}{3}} + \frac{3}{2}x^2 + c$

$x = \cos^3 t$

$= \cos t \cdot \cos^2 t$

$= \cos t \cdot (1-\sin^2 t)$

$= \cos t - \sin^2 t\cos t$

when $x=0$, $\cos^3 t = 0$

$\cos t = 0$

$t = \frac{\pi}{2}$

$y = (\sin\frac{\pi}{2})^3 = 1$

$\therefore 1 = 0 + 0 + c$

$\therefore c = 1$

$\therefore y = -\frac{9}{4}x^{\frac{4}{3}} + \frac{3}{2}x^2 + 1$

(continued on next page)

$\therefore y = -\frac{9}{4}x^{\frac{4}{3}} + \frac{3}{2}x^2 + 1$

$\therefore \int_0^1 y\,dx = \int_0^1 -\frac{9}{4}x^{\frac{4}{3}} + \frac{3}{2}x^2 + 1\,dx$

$= \left[-\frac{9}{4}\times\frac{3}{7}x^{\frac{7}{3}} + \frac{3}{2}\times\frac{1}{3}x^3 + x\right]_0^1$

$= \left[-\frac{27}{28}x^{\frac{7}{3}} + \frac{1}{2}x^3 + x\right]_0^1$

$= \left[-\frac{27}{28} + \frac{1}{2} + 1 - 0 + 0 + 0\right]$

$= \frac{15}{28}$

$\therefore$ Area of the astroid is (as it's symmetrical):

$A = \frac{15}{28} \times 4 = \frac{15}{7}$

Q5

(a) $\tan 1° \times \tan 2° \times \tan 3° \times \dots \times \tan 88° \times \tan 89°$

$= \frac{\sin 1° \sin 2° \sin 3° \sin 4° \dots \sin 88° \sin 89°}{\cos 1° \cos 2° \cos 3° \cos 4° \dots \cos 88° \cos 89°}$

$\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\therefore \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$

$\therefore \sin 1° \sin 2° = \frac{\cos 1° - \cos 3°}{2}$

$\sin 3° \sin 4° = \frac{\cos 1° - \cos 7°}{2}$

$\sin 5° \sin 6° = \frac{\cos 1° - \cos 11°}{2}$

① $\cos 1° \cos 89° = \frac{\cos 90° + \cos 88°}{2}$

$= \frac{\cos 88°}{2}$

② $\cos 2° \cos 88° = \frac{\cos 90° + \cos 86°}{2} = \frac{\cos 86°}{2}$

③ $\cos 3° \cos 87° = \frac{\cos 84°}{2}$

① (∵ cosine is an even function)

$\sin 1° \sin 89° = \frac{\cos 88° - \cos 90°}{2}$

$= \frac{\cos 88°}{2}$

② $\sin 2° \sin 88° = \frac{\cos 86° - 0}{2}$

$= \frac{\cos 86°}{2}$

③ $\sin 3° \sin 87° = \frac{\cos 84° - 0}{2}$

$= \frac{\cos 84°}{2}$

$\therefore \frac{\sin 1° \sin 2° \dots \sin 88° \sin 89°}{\cos 1° \cos 2° \dots \cos 88° \cos 89°}$

$= \frac{\sin 1° \sin 89° \cdot \sin 2° \sin 88° \cdot \sin 3° \sin 87° \dots}{\cos 1° \cos 89° \cdot \cos 2° \cos 88° \cdot \cos 3° \cos 87° \dots}$

(continued at the next page ~~back~~)

$$= \frac{\frac{\cos 88^\circ}{2} \cdot \frac{\cos 86^\circ}{2} \cdot \frac{\cos 84^\circ}{2} \cdots}{\frac{\cos 88^\circ}{2} \cdot \frac{\cos 86^\circ}{2} \cdot \frac{\cos 84^\circ}{2} \cdots}$$

$$= 1.$$

(Because when we group $\sin A$ and $\sin B$ which $A+B=90^\circ$ together.
$\cos A$ and $\cos B$

$$\sin A \sin B = \frac{\cos(A-B) - \cos 90^\circ}{2} = \frac{\cos(A-B)}{2}$$

$$\cos A \cos B = \frac{\cos(A-B) - \cos 90^\circ}{2} = \frac{\cos(A-B)}{2}$$

$\therefore \sin A \sin B = \cos A \cos B$

$\therefore$ The numerator and denominator are equal).

$$\therefore \int_0^a (\tan 1^\circ \times \tan 2^\circ \times \tan 3^\circ \times \cdots \times \tan 88^\circ \times \tan 89^\circ)\, dx$$

$$= \int_0^a 1\, dx$$

$$= [x]_0^a$$

$$= a - 0$$

$$= a$$

Q5(b) $\cancel{\int \frac{1}{\sqrt{1+x^2}}\, dx}$ (On next Page) $\cancel{\int (1+x^2)^{-\frac{1}{2}}\, dx}$

Let $\cancel{u = \sqrt{1+x^2} = (1+x^2)}$

$\cancel{u = 1+x^2}$

$\cancel{\frac{du}{dx} = 2x}$ $\cancel{du = 2x\, dx}$

$\cancel{\therefore \int \frac{1}{u^{\frac{1}{2}}}\, dx}$

$\cancel{= \int u^{-\frac{1}{2}}}$

$\cancel{\text{Let } \frac{1}{\sqrt{1+x^2}}}$

Let $u = \sqrt{1+x^2}$

$\cancel{u^2 = 1+x^2}$

$\cancel{x^2 = u^2 - 1}$

$\cancel{x = \sqrt{u^2-1}}$

$\cancel{\therefore \frac{du}{dx} \neq 2u = 1+2x}$

$\cancel{\frac{du}{dx} = \frac{1+2x}{2\sqrt{1+x^2}}}$

$\cancel{\therefore \int \frac{1}{\sqrt{1+x^2}}\, dx}$

$\cancel{= \int \frac{1}{u} \cdot \frac{1+2x}{2\sqrt{1+x^2}}\, dx}$

$\cancel{= \int \frac{1}{u} \cdot \frac{1+2\sqrt{u^2-1}}{2u}\, du}$

$\cancel{= \int \frac{1+2\sqrt{u^2-1}}{2u^2}\, du}$

Let $y=$
$\ln|\sqrt{1+x^2}+x|+c$

$y=\ln|\sqrt{1+x^2}+x|+c$

$$\frac{dy}{dx}=\frac{1\times[1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x]}{\sqrt{1+x^2}+x}=\frac{1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{\sqrt{1+x^2}+x}=\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\times\frac{1}{(\sqrt{1+x^2}+x)}$$

$$=\frac{1}{\sqrt{1+x^2}}$$

$$\therefore \frac{dy}{dx}=\frac{d(\ln|\sqrt{1+x^2}+x|+c)}{dx}=\frac{1}{\sqrt{1+x^2}}$$

$$\therefore \int\frac{1}{\sqrt{1+x^2}}dx=\ln|\sqrt{1+x^2}+x|+c$$

Q5 (c) Let $\frac{dy}{dx}=A$.

$$x\frac{d^2y}{dx^2}=\frac{v_1}{v_2}\sqrt{1+A^2}$$

$$\cancel{\frac{1}{\sqrt{1+A^2}}\frac{d^2y}{dx^2}=\frac{v_1}{v_2x}\cdot\frac{dx^2}{dy^2}}$$

$$\cancel{\int\frac{1}{\sqrt{1+A^2}}\frac{d^2y}{dx^2}dx=\int\frac{v_1}{v_2}\frac{1}{x}dx}$$

$$\int\frac{1}{\sqrt{1+A^2}}dx=\int\frac{v_1}{v_2x}\frac{dx^2}{d^2y}dx$$

$$\ln|\sqrt{1+A^2}+A|=\frac{v_1}{v_2}\int\frac{1}{x}\frac{dx^2}{dy^2}dx$$

$$\int\frac{1}{x}\frac{dx^2}{d^2y}dx=\int f'g=fg-\int fg'$$

Let $\cancel{f'=\frac{1}{x}}$, $\cancel{f=\ln x}$ Let $f'=\frac{dx^2}{d^2y}$, $f=\frac{dx}{dy}=\frac{1}{A}$

$\cancel{g=\frac{dx^2}{dy}}$, $g=\frac{1}{x}=x^{-1}$, $g'=-x^{-2}=-\frac{1}{x^2}$

$$\therefore\int\frac{1}{x}\frac{dx^2}{dy^2}dx=\frac{1}{A}\cdot\frac{1}{x}-\int\frac{1}{A}\cdot(-\frac{1}{x^2})dx$$

$$=\frac{1}{Ax}+\int\frac{1}{Ax^2}dx$$

$$=\frac{1}{Ax}+\frac{1}{A}\int\frac{1}{x^2}dx$$

$$=\frac{1}{Ax}+\frac{1}{A}(-2x^{-3})+c$$

$$=\frac{1}{Ax}-\frac{2}{Ax^3}+c$$

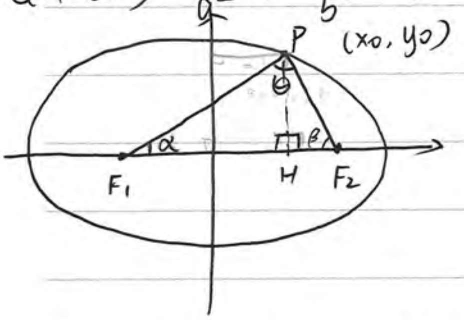
$\therefore \ln|\sqrt{1+A^2}+A|=\frac{v_1}{v_2}(\frac{1}{Ax}-\frac{2}{Ax^3})+c$

when $x=1$, $\frac{dy}{dx}=A=0$, $y=0$

$\therefore \ln|1|=\frac{v_1}{v_2}$

(Questions continue on next page)

Q4 (c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Let $\angle PF_1F_2=\alpha$, Draw $PH \perp$ x-axis

$\angle PF_2F_1=\beta$, $\angle PHF_1=\angle PHF_2=90^\circ$

$\therefore \alpha+\beta+\theta=\pi$

$HF_1=c+x_0$, $PH=y_0$,

$HF_2=c-x_0$

$\therefore F_1F_2=2c$ Area of triangle $=\frac{1}{2}\cdot 2c\cdot y_0$

$=c\cdot y_0$

93202A

Subject	Scholarship Calculus	*Standard*	93202	*Total score*	23
Q	***Score***	***Annotation***			
1	8	The candidate demonstrated a thorough understanding of 'modulus of complex numbers' in 1a and showed ability in expressing the general solutions of trig trig functions in 1b. They should show more working in finding the final solution in 1c.			
2	5	The candidate successfully applied the Vieta theorem and related the product of roots and the coefficients of quadratics in an inequality but failed to solve it correctly in 2a. They showed ability in using sine rule and double angle trig identities in 2b but failed to identify the triangle is an obtuse triangle.			
3	4	The candidate were able to construct correct mathematical models and use them to find the related rates of change problems in 3b, although made an error in dealing one of the expressions. They showed understanding of similar triangles and Pythagoras in the geometry question in 3c and could have solved it if they could apply calculus skills of optimising a function through differentiation.			
4	0	The candidate tried to use compound angle formula in 4a but abandoned their work – they could have been successful if they persevered. They could have used the results in 4a if they were able to express the integrand correctly in 4b.			
5	6	The candidate managed to prove the reciprocal relationship between tangent and cotangent functions in 5a. They showed in-depth understanding of differentiation and integration are inverse process to each other in 5b.			