S

93202Q

932022

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2022
Calculus

Time allowed: Three hours
Total score: 40

QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions.

Each question is equally weighted.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Check that this booklet has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) If z is a complex number and $|z+a| = \sqrt{a}|z+1|$ where a is a positive real number and $a \neq 1$, find the value of $|z|$ in terms of a.

(b) Solve the following system of equations for real x and y:

$$x + y = \frac{\pi}{4} \qquad \text{(A)}$$

$$\tan x + \tan y = 1 \qquad \text{(B)}$$

(c) Given $x^4 + x^3 - 4x^2 + x + 1 = 0$ and $x < 0$, find the exact value of $x^3 + \dfrac{1}{x^3}$.

QUESTION TWO

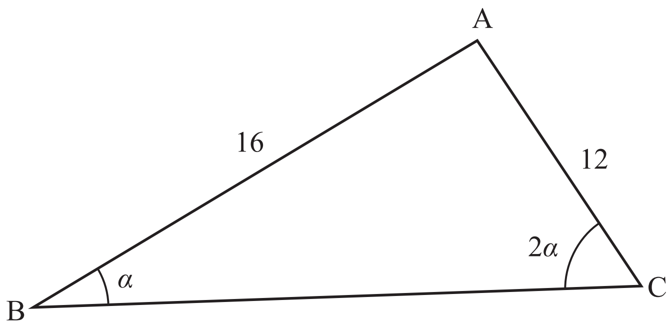
(a) Consider the equation $x^2 - 4x + 10 = k(x+1)^2$, where k is real.

For what values of k will the equation have two distinct real roots with the same sign?

(b) Consider the triangle ABC shown with:

AB = 16
AC = 12
$\angle$ABC = α
$\angle$ACB = 2α

Find the exact area of the triangle.



(c) Consider a different triangle, which has side lengths that form a geometric sequence with a common ratio of r.

If α is the angle opposite the side that is neither shortest nor longest, what is the maximum possible value of α?

Interpret your answer.

QUESTION THREE

(a) Given $f(x)=\dfrac{\mathrm{e}^{5x}\times\sqrt{x+1}}{\mathrm{e}^{\sqrt{x+1}}}$, find the exact value of $f'(0)$.

You may find the relationship $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ln f(x)\right)=\dfrac{f'(x)}{f(x)}$ helpful.

(b) A coffee filter machine can be modelled using a conical "dripper" and a cylindrical flask.

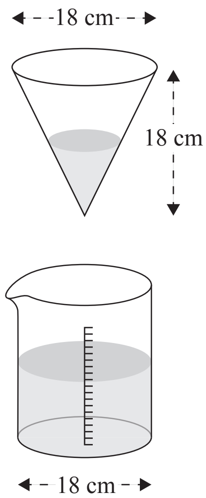
One such model is shown.

The conical dripper has a height of 18 cm, and diameter at the top of 18 cm.

The cylindrical flask has a diameter of 18 cm, and a height great enough to ensure no overflow.

Coffee drips from the dripper into the flask at a constant rate of 50 cm^3 per minute.

What is the ratio between the rate of change in depth of the coffee in the beaker and the rate of change of depth of the coffee in the dripper, when the depth in the dripper is 9 cm?



Source: www.argos.co.uk/product/914783

(c) The diagram alongside shows a sector of a circle with radius r and centre A.

$\angle BAC = 90°$

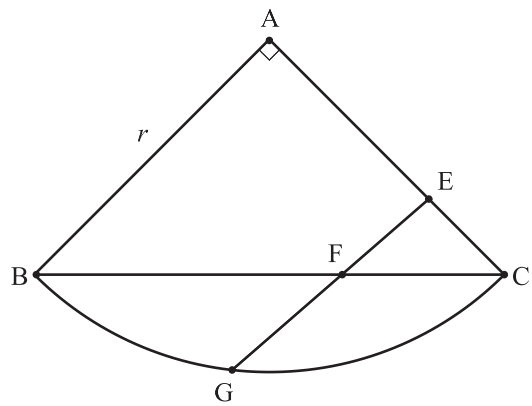
E lies on AC and G on the arc BC.

GE intersects the chord BC at F.

The points E and G move so that the line EG remains parallel to AB at all times.

Find the maximum length of FG in terms of r.

You need not prove your solution is a maximum.



QUESTION FOUR

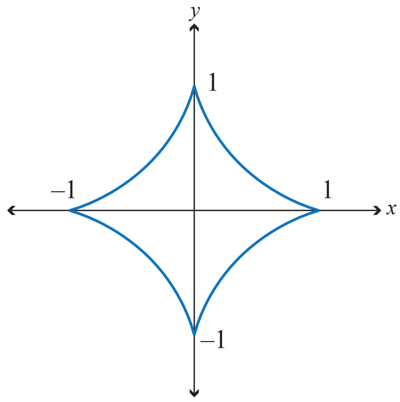
(a) It can be shown that $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$.

Use these identities, or otherwise, to show that:

$$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}.$$

(b) The astroid shown alongside is defined by the parametric equations:

$$x = \cos^3 t$$
$$y = \sin^3 t$$
$$0 \le t \le 2\pi$$

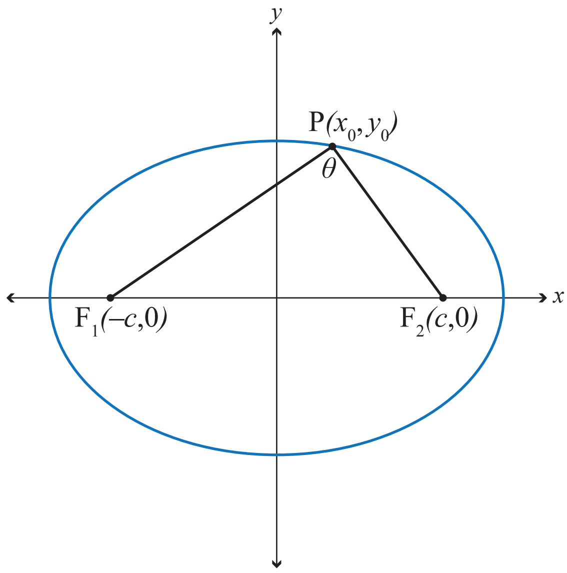


By evaluating $\int_0^1 y\,dx$ or otherwise, calculate the exact area of the astroid.

(c) The diagram alongside shows the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $a > b$.

The foci of the ellipse F_1 and F_2 have coordinates $(-c,0)$ and $(c,0)$ respectively.

P (x_0, y_0) is an arbitrary point on the ellipse such that $\angle F_1PF_2 = \theta$, as shown.



Show that the area of triangle $F_1PF_2 = b^2 \tan\dfrac{\theta}{2}$.

QUESTION FIVE

(a) Determine the value of the following definite integral in terms of a:

$$\int_0^a \left(\tan 1° \times \tan 2° \times \tan 3° \times \ldots \times \tan 88° \times \tan 89°\right) \mathrm{d}x$$

Express your answer in simplest form.

(b) Show that:

$$\int \frac{1}{\sqrt{1+x^2}} \, \mathrm{d}x = \ln\left|\sqrt{1+x^2} + x\right| + c$$

(c) Predators must engage a strategy to catch their prey. When a predator pursues its prey, the difference in their speeds will influence the path the predator takes in its pursuit.

The path $y = f(x)$ in the xy-plane that one such predator travelling at speed v_2 takes when pursuing prey travelling at speed v_1 can be modelled by the differential equation:

$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{v_1}{v_2}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

Source: www.reference.com/pets-animals/animals-predators-zebra-3784d549689c1b89

For this question you may assume that v_1 and v_2 are constant.

If $f'(1) = f(1) = 0$, find the equation of the predator's path for two distinct situations:

- $v_2 = v_1$
- $v_2 \neq v_1$

Hints: You might find the result from question 5(b) useful.

Consider the substitution $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{A}$.

93202Q