SUPERVISOR'S USE ONLY

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Draw a cross through the box (☒) if you have NOT written in this booklet

Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Scholarship 2024
Calculus

Time allowed: Three hours
Total score: 32

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should answer ALL the questions in this booklet.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Correct answers only will not be sufficient.

Check that this booklet has pages 2–24 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area will be cut off when the booklet is marked.

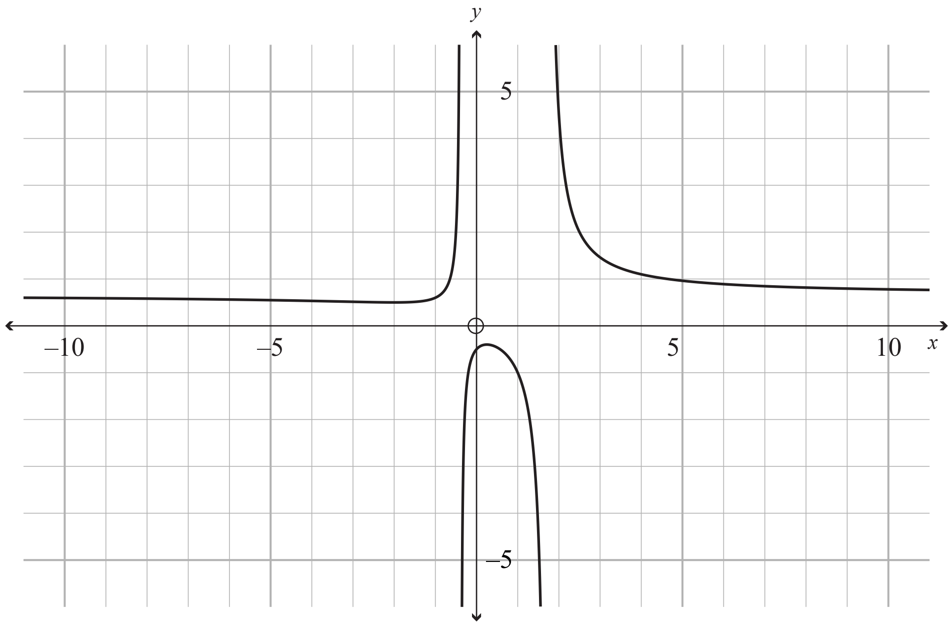
YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
TOTAL	

ASSESSOR'S USE ONLY

QUESTION ONE

(a) Consider the curve $y = \dfrac{2x^2+1}{3x^2-4x-2}$, which is shown below.



(i) Find the coordinates of any stationary points on the curve, and determine their nature.

(ii) Find the value of $\lim_{x\to\infty} \dfrac{2x^2+1}{3x^2-4x-2}$.

Hence, find the coordinate(s) where the curve intersects its own asymptote(s).

(b) For what real value(s) of α does the following system of equations have no real solutions?

$$x^3 + y^3 = 2$$

$$x + y = \alpha$$

(c) A landscaper is building steps up one side of a 48 metre high hill.

The slope of this side of the hill can be modelled by the curve

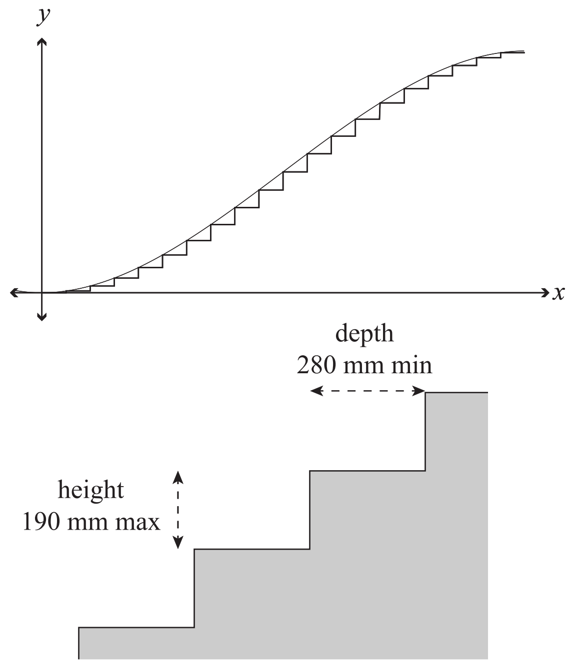
$$y = kx^2(126 - x)$$

where both x and y are in metres and k is a constant to be determined.

Building regulations state that each step needs to have a **minimum** depth of 280 mm and a **maximum** height of 190 mm, as shown.

If the landscaper chooses to build each step with a depth of 280 mm, would the steps all satisfy the maximum height regulations?

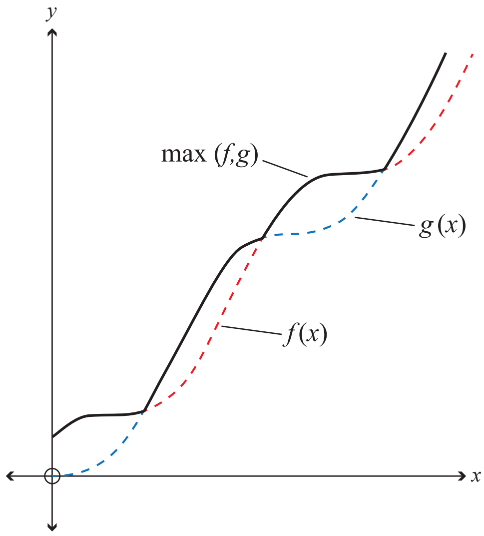
You must use calculus to support your answer.



QUESTION TWO

(a) For any two functions f and g defined on an interval, $a \le x \le b$, the **max value** function is defined as:

$$\max(f,g) = \begin{cases} f(x) \text{ if } f(x) \ge g(x) \\ g(x) \text{ if } f(x) < g(x) \end{cases}$$



To demonstrate, an example is shown opposite.

The max value function can be applied to more than two functions in a similar manner.

Consider the function $h(x) = \max\left(2\sqrt{x}, 2x, x^2\right)$ on the interval $0 \le x \le 3$.

Evaluate the definite integral $\int_0^3 h(x)\,\mathrm{d}x$.

(b) Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$, find the value of M for which $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = M\left(1+y^2\right)\left(1+3y^2\right)$.

(c) For any real number x, the **absolute value** of x is defined as:

$$|x| = \begin{cases} -x \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$$

Consider the function $s_n(x) = |x-1| + |x-2| + ... + |x-n|$ with $n > 2$.

(i) Find $s_3{}'\left(\frac{5}{2}\right)$, the derivative of $s_3(x)$ evaluated at $x = \frac{5}{2}$.

(ii) Find the minimum value of $s_{2024}(x)$.

Justify your answer.

QUESTION THREE

(a) Find the exact value of $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\cos^2 x}\,\mathrm{d}x$.

(b) Alice takes a ride at an amusement park that involves

- five spinning teacups that rotate on circular saucers
- a circular platform that rotates around a large teapot at its centre.

The coordinates of Alice's motion in the xy-plane (shown as point A in the diagram below) can be described by the parametric equations:

$$x = 4\cos\theta + \cos 4\theta$$
$$y = 4\sin\theta + \sin 4\theta$$

where θ is the anti-clockwise angle shown.

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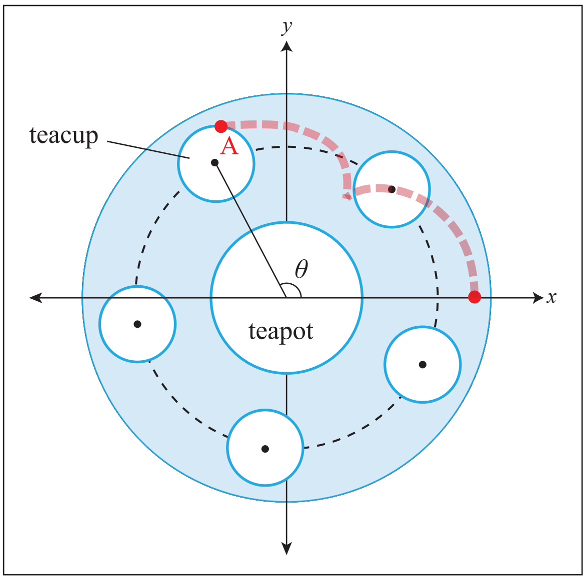


(i) Find the coordinates in the first quadrant where Alice is closest to the teapot.

Justify your answer.

(ii) Find the distance Alice travels in one complete rotation around the teapot.

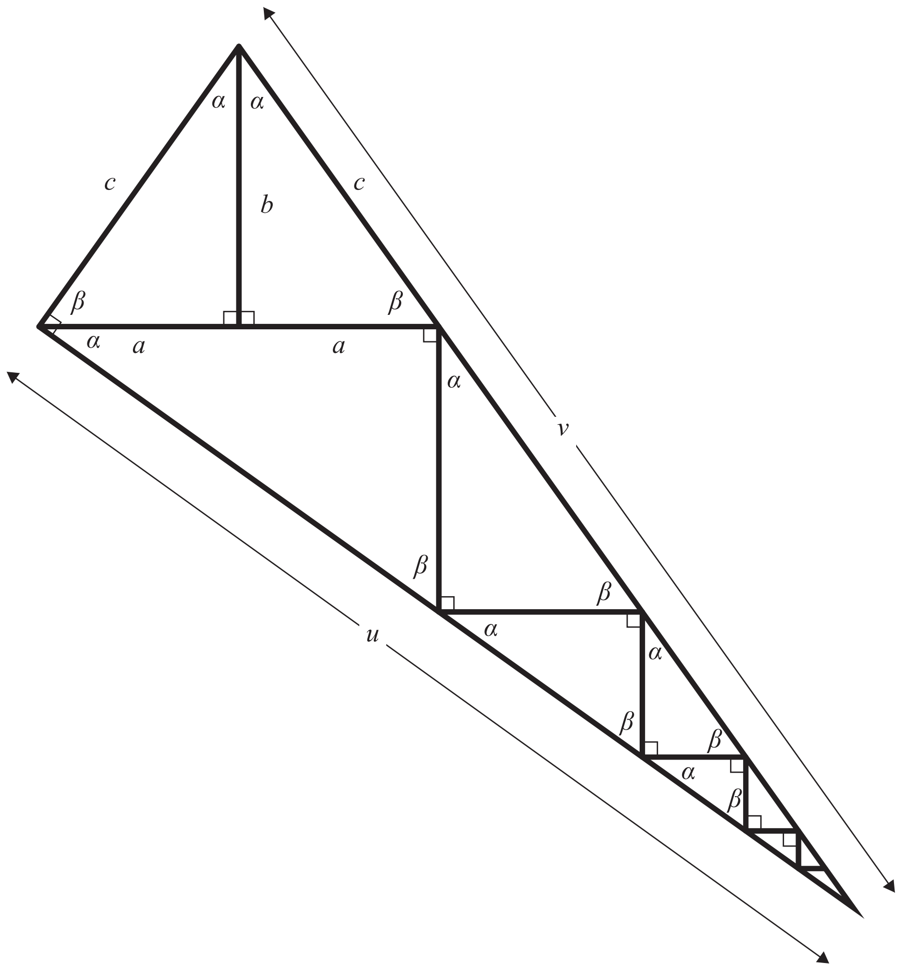
Note: when a curve is defined parametrically by the equations $x = f(\theta)$ and $y = g(\theta)$ on an interval $\alpha \leq \theta \leq \beta$, we can find its length, L, by using the formula:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta$$

(c) In 2023, two American high-school students discovered a new proof of the Pythagorean Theorem using trigonometry. In this question we will work through the key steps to derive their result.

The students' proof makes use of the diagram below, which consists of an infinite number of **similar** right-angled triangles enclosed within a large right-angled triangle. It was referred to by the students as a "waffle cone".

Note: to avoid circular logic, you should **not** make use of the Pythagorean Theorem or any of the Pythagorean trigonometric identities at any step in your working for this question.



(i) Show that $c^2 = \dfrac{2ab}{\sin 2\alpha}$.

(ii) Show that $u = \dfrac{2abc}{b^2 - a^2}$ and find a similar expression for v.

Hence, prove that $a^2 + b^2 = c^2$.

Hint: the following formulae will prove useful:

$$T_n = T_1 r^{n-1} \qquad S_n = T_1\left(\frac{1-r^n}{1-r}\right)$$

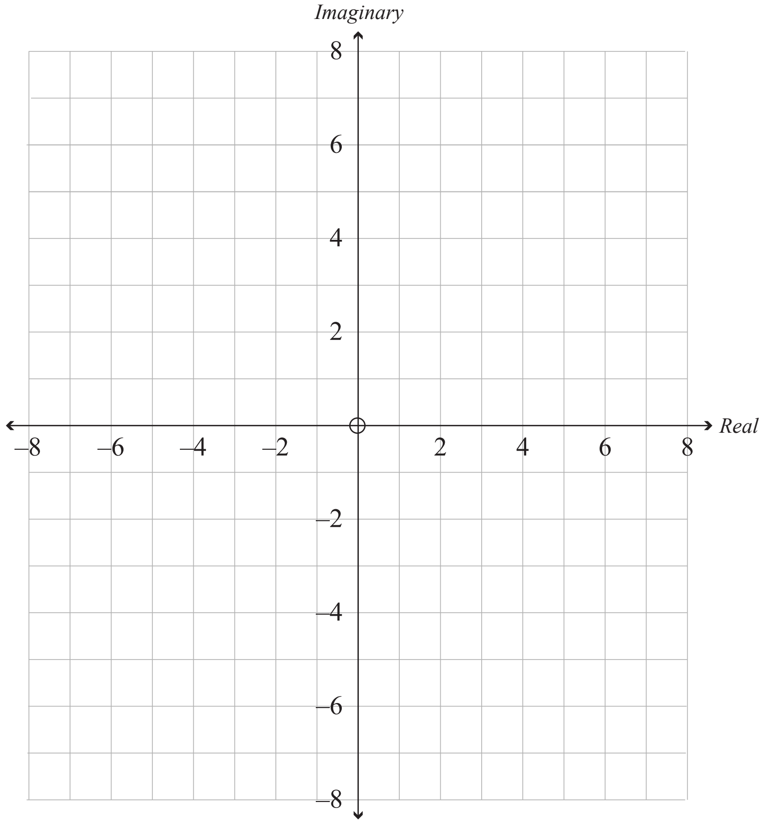
QUESTION FOUR

(a) Consider all the complex numbers z that satisfy all of the following three conditions:

- $-\dfrac{\pi}{3} \leq \arg(z) \leq \dfrac{\pi}{3}$
- $z + \bar{z} \leq 4$
- $|z| \geq 2$

Find the exact area of the region generated in an Argand diagram by the locus of points that represent z.

Use the Argand diagram below to support your working.



(b) If $z = \cos\theta + \mathrm{i}\sin\theta$ and $0 \le \theta \le \dfrac{\pi}{2}$, prove the following:

(i) $\dfrac{1}{1 - z\cos\theta} = 1 + \mathrm{i}\cot\theta$

(ii) $2\arg(z + 1) = \arg(z)$

(c) The point P $(2p, p)$ is some point on the line $y = \frac{1}{2}x$ where $p \geq 0$.

(i) Consider the locus of points that are the **same distance** from P as they are from the **line** $y = -2x$.

Explain why this locus is a parabola AND clearly describe its key features.

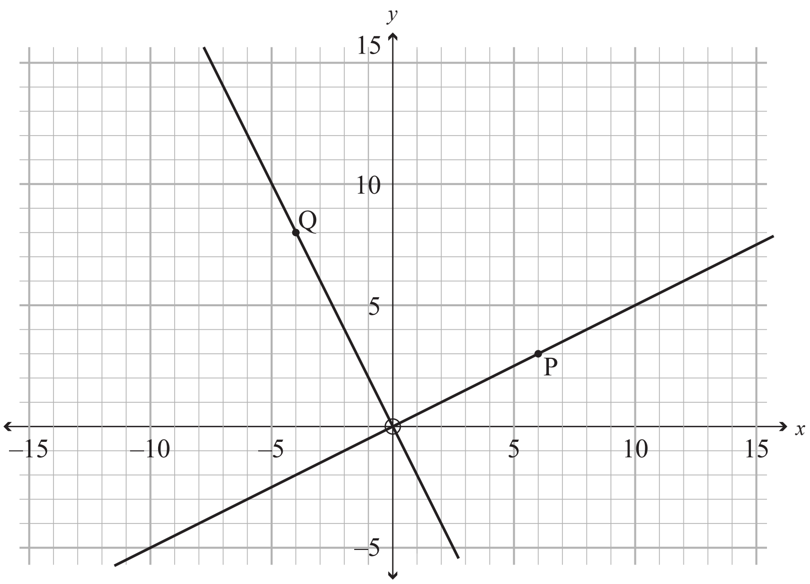
Note: you do **not** need to find the equation of the parabola.

(ii) Consider another point Q $(q, -2q)$ on the line $y = -2x$ where $q \leq 0$.

Let R be the region enclosed by the locus of points that are **three times as far** from P as they are from the **point** Q.

Now suppose the points P and Q are moving along their respective lines.

If p is increasing at a rate of 3 cm s^{-1}, and q is decreasing at a rate of 2 cm s^{-1}, at what rate is the area of the region R increasing when P is (6,3) and Q is (–4,8)?



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